

GENERALIZED AND QUASI-LOCALIZATIONS OF BRAID GROUP REPRESENTATIONS

CÉSAR GALINDO, SEUNG-MOON HONG, AND ERIC C. ROWELL

ABSTRACT. We develop a theory of localization for braid group representations associated with objects in braided fusion categories and, more generally, to Yang-Baxter operators in monoidal categories. The essential problem is to determine when a family of braid representations can be uniformly modelled upon a tensor power of a fixed vector space in such a way that the braid group generators act “locally”. Although related to the notion of (quasi-)fiber functors for fusion categories, remarkably, such localizations can exist for representations associated with objects of non-integral dimension. We conjecture that such localizations exist precisely when the object in question has dimension the square-root of an integer and prove several key special cases of the conjecture.

1. INTRODUCTION

Our aim is to generalize and develop the theory of localizations for braid group representations building upon the groundwork laid in [RW]. In this Introduction we summarize the main aspects of [RW] for the reader’s convenience, and then explain the particular achievements of the current work. We leave some relevant standard definitions to later sections for brevity’s sake.

The (n -strand) **braid group** \mathcal{B}_n is defined as the group generated by $\sigma_1, \dots, \sigma_{n-1}$ satisfying:

$$(1.1) \quad \sigma_i \sigma_j = \sigma_j \sigma_i, \quad \text{if } |i - j| \geq 2,$$

$$(1.2) \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}.$$

The following definition appears in [RW], related to the notion of *towers of algebras* found in [GHJ]:

Definition 1.1. An indexed family of complex \mathcal{B}_n -representations (ρ_n, V_n) is a **sequence of braid representations** if there exist injective algebra homomorphisms $\tau_n : \mathbb{C}\rho_n(\mathcal{B}_n) \rightarrow \mathbb{C}\rho_{n+1}(\mathcal{B}_{n+1})$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}\mathcal{B}_n & \longrightarrow & \mathbb{C}\rho_n(\mathcal{B}_n) \\ \downarrow \iota & & \downarrow \tau_n \\ \mathbb{C}\mathcal{B}_{n+1} & \longrightarrow & \mathbb{C}\rho_{n+1}(\mathcal{B}_{n+1}) \end{array}$$

The last author is partially supported by NSA grant H98230-10-1-0215.

where $\iota : \mathbb{C}\mathcal{B}_n \rightarrow \mathbb{C}\mathcal{B}_{n+1}$ is the homomorphism given by $\iota(\sigma_i) = \sigma_i$.

Usually the homomorphisms τ_n will be clear from the context and will be suppressed. Examples of sequences of braid representations of interest include those obtained from specialized quotients of $\mathbb{C}(q)\mathcal{B}_n$ (e.g. Temperley-Lieb algebras [J1]) and from objects in braided fusion categories. An important role is played by the basic class of examples obtained from *braided vector spaces* (W, R) : that is, a vector space W and an automorphism $R \in \text{End}(W^{\otimes 2})$ satisfying $(I \otimes R)(R \otimes I)(I \otimes R) = (R \otimes I)(I \otimes R)(R \otimes I)$. These sequences of \mathcal{B}_n -representations will be denoted $(\rho^{(W,R)}, W^{\otimes n})$.

The question considered in [RW] is: when can a given *unitary* sequence of braid group representations be related to a sequence of braid group representations of the form $(\rho^{(W,R)}, W^{\otimes n})$ in the following sense:

Definition 1.2. Suppose (ρ_n, V_n) is a sequence of braid representations. A **localization** of (ρ_n, V_n) is a braided vector space (W, R) such that for all $n \geq 2$ there exist injective algebra homomorphisms $\phi_n : \mathbb{C}\rho(\mathcal{B}_n) \rightarrow \text{End}(W^{\otimes n})$ such that $\phi_n \circ \rho = \rho^{(W,R)}$.

If (ρ_n, V_n) are unitary representations and $R \in \text{U}(W^{\otimes 2})$ we say that (W, R) is a **unitary localization**.

The sequences of braid representations studied in [RW] are constructed as follows: Let X be an object in a braided fusion category \mathcal{C} . The braiding c on \mathcal{C} induces algebra homomorphisms $\psi_X^n : \mathbb{C}\mathcal{B}_n \rightarrow \text{End}_{\mathcal{C}}(X^{\otimes n})$ via $\sigma_i \rightarrow I_X^{\otimes i-1} \otimes c_{X,X} \otimes I_X^{\otimes n-i-1}$ (where we have suppressed the associativities for notational convenience). The left action of $\text{End}_{\mathcal{C}}(X^{\otimes n})$ on the minimal faithful module $W_n^X := \bigoplus_i \text{Hom}_{\mathcal{C}}(X_i, X^{\otimes n})$ (X_i simple subobjects of $X^{\otimes n}$) yields a sequence of braid representations (ρ_X, W_n^X) . A main result is the following:

Proposition 1.3 (see [RW] Theorem 4.5). *Suppose that X is a simple object in a braided fusion category \mathcal{C} , the sequence (ρ_X, W_n^X) is localizable and ψ_X^n is surjective (so that $\text{Hom}_{\mathcal{C}}(X_i, X^{\otimes n})$ is irreducible as a \mathcal{B}_n representation for each i). Then $\text{FPdim}(X)^2 \in \mathbb{N}$.*

Otherwise stated, the hypotheses imply that the fusion subcategory generated by X is *weakly integral*.

On the other hand, the sequence of braid group representations (ρ_X, W_n^X) associated with an object $X \in \text{Rep}(H)$ for a finite dimensional semisimple quasi-triangular Hopf algebra H is easily seen to be localizable (see Prop. 4.17 for a stronger statement). Notice that $\text{Rep}(H)$ is an *integral* fusion category in this case (*i.e.* $\text{FPdim}(X) \in \mathbb{N}$ for all $X \in \text{Rep}(H)$). The main conjecture in [RW] is the following:

Conjecture 1.4 (cf. [RW] Conjecture 4.1). *Let $X \in \mathcal{C}$ be a simple object in a braided fusion category. Then (ρ_X, W_n^X) is localizable if, and only if, $\text{FPdim}(X)^2 \in \mathbb{N}$.*

A related conjecture is the following:

Conjecture 1.5 (cf. [RSW] Conjecture 6.6). *Let X be a simple object in a braided fusion category. Then the image of \mathcal{B}_n under (ρ_X, W_n^X) is a finite group if, and only if, $\text{FPdim}(X)^2 \in \mathbb{N}$.*

The following brings these ideas full circle:

Conjecture 1.6 (cf. [RW] Conjecture 3.1). *Suppose (V, R) is a braided vector space such that R is unitary and finite order. Then the image of the \mathcal{B}_n representation on $V^{\otimes n}$ defined by $\sigma_i \rightarrow I_V^{\otimes i-1} \otimes R \otimes I_V^{\otimes n-i-1}$ has finite image.*

It should be noted that neither the unitarity nor finite order condition can be dropped.

The following is a summary of the current work: In Section 3 we generalize the notion of Yang-Baxter operators in two ways, leading to *quasi-* and (k, m) -*generalized* braided vector spaces. We give a more flexible version of Definition 1.1 in terms of sequences of algebras equipped with braid group representations (Definition 4.5). In Section 4 we define \mathcal{C} -localizations over arbitrary monoidal categories \mathcal{C} , so that a Vec_f -localization is the same as Definition 1.2. Moreover, when \mathcal{C} is a monoidal category associated with a quasi-braided vector space we obtain the notion of a *quasi-localization* and prove that modules over quasitriangular *quasi-Hopf* algebras lead to *quasi-localizable* sequences of algebras just as modules over quasitriangular Hopf algebras lead to localizable sequences of algebras. As a by-product we obtain a criterion for the existence of a fiber functor for an integral braided fusion category. To continue the analogy with reconstruction-type theorems we describe *weak* localizations as well. A variant of localization associated with (k, m) -generalized braided vector spaces is also given, which, although somewhat mysterious, is more explicit than quasi-localizations. The main result of the somewhat technical Section 5 is that the braid group representations associated with any weakly group-theoretical braided fusion category \mathcal{C} are unitarizable, and if in addition \mathcal{C} is integral a unitary (quasi-)localization exists. We also give an example associated with quantum \mathfrak{sl}_3 at 6th roots of unity which illustrates the differences between the various forms of localization studied here. In Section 6 we state a version of the conjectures mentioned above for quasi- and generalized localizations and provide evidence. In particular we prove the statement analogous to [RW, Theorem 4.5] (see above) in these settings.

Acknowledgements E.R. thanks D. Naidu and Z. Wang for useful comments.

2. PRELIMINARIES

In this section we recall some standard notions in order to establish notation and conventions. Much of the material here can be found in [BK], but we typically adopt the conventions of [ENO1].

2.1. k -linear categories. Let k be a field. A k -linear category \mathcal{C} is a category in which the Hom-sets are k -vector spaces, the compositions are k -bilinear (we do not assume the existence of direct sums or zero object, so \mathcal{C} may not be additive). The notion of a k -linear functor $\mathcal{C} \rightarrow \mathcal{D}$, and a k -bilinear bifunctor $\mathcal{C} \times \mathcal{C}' \rightarrow \mathcal{D}$ for k -linear categories $\mathcal{C}, \mathcal{C}', \mathcal{D}$, will be obvious.

A k -linear category \mathcal{C} is said to be of **locally finite dimension** if $\text{Hom}_{\mathcal{C}}(X, Y)$ is of finite dimension for any $X, Y \in \mathcal{C}$. In this paper we shall only consider k -linear categories of locally finite dimension.

2.2. C^* -categories. Most of the material here can be found in [Mu1].

Definition 2.1. A \mathbb{C} -linear category \mathcal{D} is called a **complex $*$ -category** if:

- (1) There is an involutive antilinear contravariant endofunctor $*$ of \mathcal{D} which is the identity on objects. The image of f under $*$ will be denoted by f^* .
- (2) For each $f \in \text{Hom}_{\mathcal{D}}(X, Y)$, $f^*f = 0$ implies $f = 0$.

In particular, in a complex $*$ -category, each $\text{Hom}_{\mathcal{D}}(X, X)$ is a $*$ -algebra with identity. A **C^* -category** \mathcal{D} is a complex $*$ -category such that the spaces $\text{Hom}_{\mathcal{D}}(X, Y)$ are Banach spaces and the norms satisfy

$$\|fg\| \leq \|f\| \|g\|, \quad \|f^*f\| = \|f\|^2,$$

for all $f \in \text{Hom}_{\mathcal{D}}(X, Y), g \in \text{Hom}_{\mathcal{D}}(Y, Z)$. (Then the algebras $\text{Hom}_{\mathcal{D}}(X, X)$ are C^* -algebras.)

Remark 2.2. Every abelian complex $*$ -category of locally finite dimension admits a unique structure of C^* -category (see [Mu1, Proposition 2.1]). Since we are only interested in categories of locally finite dimension, for us abelian C^* -category and abelian complex $*$ -category are equivalent notions.

Example 2.3. Let R be a finite dimensional C^* -algebra, then the category $\mathcal{U}\text{-Rep}(R)$ of $*$ -representations on finite dimensional Hilbert spaces is an abelian $*$ -category and thus admits a unique C^* -structure. Note $\text{Rep}(R)$ is equivalent to $\mathcal{U}\text{-Rep}(R)$.

Let X and Y be objects in a $*$ -category. A morphism $u : X \rightarrow Y$ is **unitary** if $uu^* = I_Y$ and $u^*u = I_X$. A morphism $a : X \rightarrow X$ is **self-adjoint** if $a^* = a$.

A natural transformation $\gamma : F \rightarrow G$, between functors $F, G : \mathcal{D}_1 \rightarrow \mathcal{D}_2$ with \mathcal{D}_2 a $*$ -category is called **unitary natural transformation** if γ_X is unitary for each $X \in \mathcal{D}_1$.

Remark 2.4. Let \mathcal{D} a $*$ -category

- (1) The opposite category \mathcal{D}^{op} is a $*$ -category with the same $*$ -structure.
- (2) Every isomorphism in a C^* -category has a polar decomposition, *i.e.*, if $f : X \rightarrow Y$ is an isomorphism, then $f = ua$ where $a : X \rightarrow X$ is self-adjoint and $u : X \rightarrow Y$ is unitary, see [B, Proposition 8].

Definition 2.5. A ***-functor** $F : \mathcal{D} \rightarrow \mathcal{D}'$ between *-categories \mathcal{D} and \mathcal{D}' is \mathbb{C} -linear functor such that $F(f^*) = F(f)^*$ for all $f \in \text{Hom}_{\mathcal{D}}(X, Y)$.

Remark 2.6. Let R be a finite dimensional C^* -algebra, then every exact endofunctor $F : \mathcal{U}\text{-Rep}(R) \rightarrow \mathcal{U}\text{-Rep}(R)$ is naturally equivalent to a functor of the form $M \otimes_R (?)$, where $M \in \text{Bimod}(R)$ is an R -bimodule. The functor $M \otimes_R (?)$ is a *-functor if and only if M is a unitary R -bimodule or equivalently a unitary $R \otimes R^{op}$ -module. We shall denote the *-category of unitary R -bimodules as $\mathcal{U}\text{-Bimod}(R)$.

Two *-categories \mathcal{D} and \mathcal{D}' are **unitarily equivalent** if there exist *-functors $F : \mathcal{D} \rightarrow \mathcal{D}'$ and $G : \mathcal{D}' \rightarrow \mathcal{D}$ and unitary natural isomorphisms $I_{\mathcal{D}} \rightarrow G \circ F$, $I_{\mathcal{D}'} \rightarrow F \circ G$.

Remark 2.7. (1) Every equivalence of *-category is equivalent to a unitary equivalence.

(2) Every abelian *-category is unitary equivalent to a category $\bigoplus_{i \in I} \text{Hilb}_f$, a direct sum of copies of the category of finite-dimensional Hilbert spaces.

Given *-categories \mathcal{D}_1 and \mathcal{D}_2 we define the \mathbb{C} -linear category $\text{Hom}^*(\mathcal{D}_1, \mathcal{D}_2)$, where objects are *-functors $F : \mathcal{D}_1 \rightarrow \mathcal{D}_2$ such that $F(X) \neq 0$ for finite isomorphism classes of simple objects, and morphisms are natural transformations. The category $\text{Hom}^*(\mathcal{D}_1, \mathcal{D}_2)$ has a natural structure of *-category with *-structure $(\alpha^*)_X = (\alpha_X)^*$.

2.3. Unitary fusion categories and module categories. For us, a monoidal category $(\mathcal{C}, \otimes, \alpha, 1, \lambda, \rho)$ will always be k -linear with associativity constraint $\alpha_{V,W,Z} : (V \otimes W) \otimes Z \rightarrow V \otimes (W \otimes Z)$, unit object 1 and unit constraints λ, ρ satisfying the usual (triangle and pentagon) axioms. Note that we do not assume that 1 is a simple object. As is customary, we will assume that the unit constraints are identities and abuse notation by referring to “the monoidal category $(\mathcal{C}, \otimes, \alpha)$.”

Definition 2.8. A **multi-fusion category** is a monoidal, rigid, semisimple category with a finite number of isomorphism classes of simple objects. A **fusion category** is a multi-fusion category in which 1 is a simple object.

A **unitary (multi)-fusion category** is a (multi)-fusion category $(\mathcal{C}, \otimes, \alpha)$, where \mathcal{C} is a positive *-category, the constraints are unitary natural transformations, and $(f \otimes g)^* = f^* \otimes g^*$, for every pair of morphisms f, g in \mathcal{C} .

Example 2.9. (1) The category of finite dimensional Hilbert spaces Hilb_f , with the tensor product of Hilbert spaces is a unitary fusion category.

(2) If R is a finite dimensional C^* -algebra, then $\mathcal{U}\text{-Bimod}(R)$ is a unitary multi-fusion category.

(3) Recall that a finite dimensional (quasi) Kac algebra is a (quasi) Hopf algebra H , such that H is a C^* -algebra, Δ and ε are *-algebras morphisms,

and if H is a quasi-Hopf algebra the associator must satisfy $\Phi^* = \Phi^{-1}$. In this case the category of unitary H -modules is a unitary fusion category.

A ***-monoidal functor** between unitary fusion categories is a monoidal functor $(F, F^0, F^1) : \mathcal{C}_1 \rightarrow \mathcal{C}_2$, such that F is a *-functor, and $F_{X,Y}^0, F_X^1$ are unitary natural transformations.

Module categories over monoidal categories are defined in [O1].

Definition 2.10. Let \mathcal{C} be a unitary fusion category. A left **\mathcal{C} -module *-category** is a left \mathcal{C} -module category $(\mathcal{M}, \overline{\otimes}, \mu)$ such that \mathcal{M} is a *-category, the constraints are unitary natural transformations, and $(f \overline{\otimes} g)^* = f^* \overline{\otimes} g^*$ for all $f \in \mathcal{C}, g \in \mathcal{M}$.

Definition 2.11. A **\mathcal{C} -module *-functor** $F : \mathcal{M} \rightarrow \mathcal{N}$ between \mathcal{C} -module *-categories \mathcal{M}, \mathcal{N} , is \mathcal{C} -module functor (F, F^0, F^1) , such that F is a *-functor, and F^0, F^1 are unitary natural transformations.

Recall that a monoidal category is called strict if the associativity constraint is the identity. A module category $(\mathcal{M}, \overline{\otimes}, \mu)$ over a strict monoidal category is called a **strict module category** if the constraint μ is the identity. Using the same argument as in [Ga1, Proposition 2.2] we may assume that every unitary fusion category \mathcal{C} and every \mathcal{C} -module *-category is strict.

If $\alpha : F \rightarrow G$ is a module natural transformation between \mathcal{C} -module *-functors, $\alpha^* : G \rightarrow F$ is a \mathcal{C} -module natural transformation. Thus the category $\text{Hom}_{\mathcal{C}}^*(\mathcal{M}, \mathcal{N})$ of all \mathcal{C} -module *-functors and \mathcal{C} -module natural transformations has a *-structure.

3. QUASI- AND GENERALIZED YANG-BAXTER OPERATORS

3.1. Yang-Baxter operators. We shall recall the definition of Yang-Baxter operator on a monoidal category, see [Ks]:

Definition 3.1. If V is an object of a monoidal category $(\mathcal{C}, \otimes, \alpha)$ and $c \in \text{Aut}(V \otimes V)$ satisfies the equation

$$(3.1) \quad \alpha_{V,V,V}(c \otimes I) \alpha_{V,V,V}^{-1}(I \otimes c) \alpha_{V,V,V}(c \otimes I) = (I \otimes c) \alpha_{V,V,V}(c \otimes I) \alpha_{V,V,V}^{-1}(I \otimes c) \alpha_{V,V,V}$$

(where $I = I_V$) then c is called a **Yang-Baxter operator** on V .

Yang-Baxter operators define representations of the braid groups in the following way: define $V^{\otimes 1} = V$, $V^{\otimes n} = V^{\otimes(n-1)} \otimes V$ that is, all left parentheses appear left of the first V with the same convention for tensor products of morphisms. Define automorphisms c_1, \dots, c_{n-1} of $V^{\otimes n}$ by

$$(3.2) \quad c_i = (\alpha_{V^{\otimes(i-1)}, V, V}^{-1} \otimes I_V^{\otimes(n-i-1)})(I_{V^{\otimes(i-1)}} \otimes c \otimes I_V^{\otimes(n-i-1)})(\alpha_{V^{\otimes(i-1)}, V, V} \otimes I_V^{\otimes(n-i-1)}),$$

where, for example,

$$\alpha_{V^{\otimes 3}, V, V} \otimes I_V^{\otimes 2} = (\alpha_{V^{\otimes 3}, V, V} \otimes I_V) \otimes I_V \in \text{Hom}_{\mathcal{C}}(V^{\otimes 7}, ((V^{\otimes 3} \otimes (V \otimes V)) \otimes V) \otimes V).$$

Thus, for any n the map

$$(3.3) \quad \begin{aligned} \rho_n : \mathcal{B}_n &\rightarrow \text{Aut}_{\mathcal{C}}(V^{\otimes n}) \\ \sigma_i &\mapsto c_i \end{aligned}$$

is a group homomorphism (see [Ks, Lemma XV.4.1]).

Definition 3.2. A Yang-Baxter operator c on a finite dimensional vector space $V \in \text{Vec}_f$ is called a **braided vector space** (V, c) . A **unitary** braided vector space (\mathcal{H}, c) is a unitary Yang-Baxter operator c on an object \mathcal{H} in the category of finite dimensional Hilbert spaces.

Remark 3.3. The group $\text{Aut}_{\mathcal{C}}(V^{\otimes n})$ is linear: it has a faithful action on the vector space $\text{End}_{\mathcal{C}}(V^{\otimes n})$. Pulling back via ρ_n we obtain a linear representation of \mathcal{B}_n on $\text{End}_{\mathcal{C}}(V^{\otimes n})$.

In the special case of braided vector spaces (V, c) (*i.e.* $\mathcal{C} = \text{Vec}_f$) one has $\text{Aut}_{\mathcal{C}}(V^{\otimes n}) = GL(V^{\otimes n})$ so that one obtains a linear representation of \mathcal{B}_n on $V^{\otimes n}$. Moreover, the associativity isomorphisms for Vec_f are: $\alpha_{V,V,V} : (v_1 \otimes v_2) \otimes v_3 \rightarrow v_1 \otimes (v_2 \otimes v_3)$ so that, with respect to the obvious compatible choices of bases, the $\alpha_{V^{\otimes(i-1)}, V, V}$ are all represented by the identity matrix. Thus the \mathcal{B}_n -representation on $V^{\otimes n}$ obtained from ρ_n is equivalent to a *matrix* representation of the form

$$\sigma_i \rightarrow I^{\otimes i-1} \otimes c \otimes I^{\otimes n-i-1}.$$

In particular, our definition of braided vector space is the same as that of [AS], *i.e.* a pair (V, c) where $c \in \text{Aut}(V \otimes V)$ satisfies (3.1) on $V^{\otimes 3}$ with the α removed.

3.2. Quasi Yang-Baxter operators. Let (\mathcal{D}, \otimes) be a monoidal category and $A \in \mathcal{D}$ an object. We shall denote by $\langle A \rangle$ the monoidal subcategory of \mathcal{D} generated by A , that is, the objects of $\langle A \rangle$ are isomorphism classes of $A^{\otimes n}$ for $n \geq 0$ ($A^{\otimes 0} = 1$).

Let $a = \{a_{X,Y}\}_{X,Y \in \langle A \rangle}$ be a family of isomorphisms $a_{X,Y} : (X \otimes A) \otimes Y \rightarrow X \otimes (A \otimes Y)$, $X, Y \in \langle A \rangle$. We shall say that a is **self-natural** if $a_{X,Y}$ is natural in X and Y for every pair of isomorphisms constructed as a composition of tensor products of identities, elements and inverses of elements in a .

If $a = \{a_{X,Y}\}_{X,Y \in \langle A \rangle}$ is a self-natural family of isomorphisms we define $a^2 = \{a_{X,Y}^2\}_{X,Y \in \langle A \rangle}$ a new family of self-natural on $A \otimes A$ by the commutativity of pentagonal diagram:

$$\begin{array}{ccc}
& (X \otimes A) \otimes (A \otimes Y) & \\
a_{X \otimes A, Y} \nearrow & & \searrow a_{X, A \otimes Y} \\
((X \otimes A) \otimes A) \otimes Y & & X \otimes (A \otimes (A \otimes Y)) \\
a_{X, A \otimes I_Y} \searrow & & \nearrow I_X \otimes a_{A, Y} \\
(X \otimes (A \otimes A)) \otimes Y & \xrightarrow{a_{X, Y}^2} & X \otimes ((A \otimes A) \otimes Y)
\end{array}$$

Definition 3.4. Let \mathcal{D} be a monoidal category and A an object in \mathcal{D} . A **quasi-Yang-Baxter operator** on A is a pair (a, c) , where $a = \{a_{X, Y}\}_{X, Y \in \langle A \rangle}$ is a family of self-natural transformations and $c : A \otimes A \rightarrow A \otimes A$ is an automorphism such that

$$(3.4) \quad a_{A, A}(c \otimes I)a_{A, A}^{-1}(I \otimes c)a_{A, A}(c \otimes I) = (I \otimes c)a_{A, A}(c \otimes I)a_{A, A}^{-1}(I \otimes c)a_{A, A},$$

and the diagram

$$(3.5) \quad \begin{array}{ccc}
(X \otimes (A \otimes A)) \otimes Y & \xrightarrow{a_{X, Y}^2} & X \otimes ((A \otimes A) \otimes Y) \\
\downarrow (I_X \otimes c) \otimes I_Y & & \downarrow I_X \otimes (c \otimes I_Y) \\
(X \otimes (A \otimes A)) \otimes Y & \xrightarrow{a_{X, Y}^2} & X \otimes ((A \otimes A) \otimes Y)
\end{array}$$

commutes for all $X, Y \in \langle A \rangle$.

If \mathcal{D} is a C^* -tensor category and A is an object, a **unitary quasi-Yang-Baxter operator** on A is a quasi-Yang-Baxter operator (a, c) such that c and $a_{X, Y}$ are unitary isomorphisms for all $X, Y \in \langle A \rangle$.

Example 3.5.

Quasi-Yang-Baxter operators generalize the notion of Yang-Baxter operators over monoidal categories. In fact, if (A, c) is a Yang-Baxter operator over a monoidal category $(\mathcal{C}, \otimes, \alpha)$ then $a_{X, Y} = \alpha_{X, A, Y}$ defines a self-natural family of isomorphisms such that $a_{X, Y}^2 = \alpha_{X, A \otimes A, Y}$ (by the pentagonal identity), and by the naturality of α , the diagram 3.5 commutes, so $(\alpha_{X, A, Y}, c)$ is a quasi-Yang-Baxter operator on A .

Definition 3.6. A **quasi-braided vector space** is a quasi-Yang-Baxter operator over Vec_f . A **unitary quasi-braided vector space** is a unitary quasi-Yang-Baxter operator in the category of finite dimensional Hilbert spaces.

Example 3.7. Let (H, Φ, R) be a quasi-triangular quasi-Hopf algebra and V an H -module. Then for each pair $X, Y \in \langle V \rangle$ the family of isomorphisms $a_{X,Y}((x \otimes v) \otimes y) = \Phi(x \otimes (v \otimes y))$ is self-natural and $c : V \otimes V \rightarrow V \otimes V$ given by $c(v \otimes v') = (R(v \otimes v'))_{21}$ defines a quasi-braided vector space structure on $V \in \text{Vec}_f$. (recall that $(v \otimes w)_{21} = w \otimes v$). Observe that c is a Yang-Baxter operator on $V \in \text{Rep}(H)$, but cannot be a braided vector space unless the forgetful functor is a fiber functor.

Let (a, c) be a quasi-Yang-Baxter operator on $A \in \mathcal{C}$. We define $A^{\otimes 1} = A$, $A^{\otimes n} = A^{\otimes(n-1)} \otimes A$ and automorphisms c_1, \dots, c_{n-1} of $A^{\otimes n}$ by

$$(3.6) \quad c_i = (a_{A^{\otimes(i-1)}, A}^{-1} \otimes I_A^{\otimes(n-i-1)})(I_{A^{\otimes(i-1)}} \otimes c \otimes I_A^{\otimes(n-i-1)})(a_{A^{\otimes(i-1)}, A} \otimes I_A^{\otimes(n-i-1)}).$$

Now suppose that \mathcal{C} is strict. We define a monoidal category $\overline{(A, a, c)}$ where $\text{Obj}(\overline{(A, a, c)}) = \text{Obj}(A)$ and the tensor product of objects is the same as in \mathcal{C} . We define inductively self-natural families of isomorphisms $a_{X,Y}^n : (X \otimes A^{\otimes n}) \otimes Y \rightarrow X \otimes (A^{\otimes n} \otimes Y)$ by $a^1 = a$ and $a_{X,Y}^n$ by the commutativity of the diagram

$$\begin{array}{ccc} & (X \otimes A^{\otimes(n-1)}) \otimes (A \otimes Y) & \\ \nearrow a_{X \otimes A^{\otimes(n-1)}, Y} & & \searrow a_{X, A \otimes Y}^{n-1} \\ ((X \otimes A^{\otimes(n-1)}) \otimes A) \otimes Y & & X \otimes (A^{\otimes(n-1)} \otimes (A \otimes Y)) \\ \searrow a_{X, A}^{n-1} \otimes I_Y & & \nearrow I_X \otimes a_{A^{\otimes(n-1)}, Y} \\ (X \otimes (A^{\otimes(n-1)} \otimes A)) \otimes Y & \xrightarrow{a_{X,Y}^n} & X \otimes ((A^{\otimes(n-1)} \otimes A) \otimes Y) \end{array}$$

A morphism $f : A^{\otimes n} \rightarrow A^{\otimes m}$ in $\overline{(A, a, c)}$ is a morphism in \mathcal{C} such that the diagram

$$\begin{array}{ccc} (X \otimes A^{\otimes n}) \otimes Y & \xrightarrow{a_{X,Y}^n} & X \otimes (A^{\otimes n} \otimes Y) \\ \downarrow (I_X \otimes f) \otimes I_Y & & \downarrow I_X \otimes (f \otimes I_Y) \\ (X \otimes A^{\otimes m}) \otimes Y & \xrightarrow{a_{X,Y}^m} & X \otimes (A^{\otimes m} \otimes Y) \end{array}$$

is commutative for any $X, Y \in \overline{(A, a, c)}$.

Using the same arguments of [Dav, Section 2.1], we can prove that $\overline{(A, a, c)}$ is a monoidal category with tensor product \otimes and associativity constraint $\alpha_{A^{\otimes m}, A^{\otimes n}, A^{\otimes s}} = a_{A^{\otimes m}, A^{\otimes s}}^n$.

Proposition 3.8. *Let (a, c) be a quasi-Yang-Baxter operator on $A \in \mathcal{C}$. There exists a unique homomorphism of groups $\rho_n : \mathcal{B}_n \rightarrow \text{Aut}_{\mathcal{C}}(A^{\otimes n})$ sending the generator σ_i of \mathcal{B}_n to c_i for each $i = 1, \dots, n-1$.*

Proof. We may assume without loss of generality that \mathcal{C} is strict, since in the non-strict case by the coherence of \mathcal{C} every object in $\langle A \rangle$ is isomorphic to $A^{\otimes n}$ by a unique isomorphism constructed as composition and tensor products of the constraint isomorphisms and their inverses. By hypothesis $c : A \otimes A \rightarrow A \otimes A$ is a Yang-Baxter operator in (A, a, c) , so the proposition follows from the group morphisms (3.3). \square

3.3. Generalized Yang-Baxter Operators. In this subsection we will describe a second approach to generalizing Yang-Baxter operators based upon the generalized Yang-Baxter equation introduced in [RZWG].

Definition 3.9. Let V be an object in a monoidal category $(\mathcal{C}, \otimes, \alpha)$ and $k, m \in \mathbb{N}$ with $k > m$. Set $I_m := I_{V^{\otimes m}}$ and denote by $\alpha_{k,m}$ the natural isomorphism $V^{\otimes k} \otimes V^{\otimes m} \rightarrow V^{\otimes m} \otimes V^{\otimes k}$ coming from α . For an automorphism $c \in \text{Aut}(V^{\otimes k})$ the (k, m) -**generalized Yang-Baxter equation** ((k, m) -**gYBE**) on V is

$$(3.7) \quad \alpha_{k,m}(c \otimes I_m) \alpha_{k,m}^{-1}(I_m \otimes c) \alpha_{k,m}(c \otimes I_m) = (I_m \otimes c) \alpha_{k,m}(c \otimes I_m) \alpha_{k,m}^{-1}(I_m \otimes c) \alpha_{k,m}.$$

A solution c to the (k, m) -gYBE is called a (k, m) -**gYB operator** on V if in addition c satisfies the following equations for all $4 \leq j$:

$$(3.8) \quad \alpha_j(c \otimes I_{m(j-2)}) \alpha_j^{-1}(I_{m(j-2)} \otimes c) = (I_{m(j-2)} \otimes c) \alpha_j(c \otimes I_{m(j-2)}) \alpha_j^{-1}.$$

where $\alpha_j = \alpha_{k,m(j-2)}$.

Remark 3.10. (1) To verify that a solution c to the (k, m) -gYBE is a (k, m) -gYB operator it is sufficient to check eqn. (3.8) for $j < \frac{k}{m} + 2$ since for $j \geq \frac{k}{m} + 2$ eqn. the operators c_1 and c_j act nontrivially on disjoint tensor factors so (3.8) is automatic.

- (2) The requirement $k > m$ is to avoid trivialities in what follows: see Prop. 3.12 below.
- (3) A Yang-Baxter operator on V in a category \mathcal{C} is a $(2, 1)$ -gYB operator.
- (4) One may also define generalized *quasi*-Yang-Baxter operators but we shall have no need to do so.

Definition 3.11. A **(unitary) (k, m) -generalized braided vector space** is a (unitary) (k, m) -gYB operator c on a finite dimensional (Hilbert) vector space $V \in \text{Vec}_f$.

As in Remark 3.3, in the case of a (k, m) -generalized braided vector space (V, c) we may suppress the α in eqns. (3.7) and (3.8) by choosing bases appropriately. Define operators

$$(3.9) \quad c_i^{k,m} = I_m^{\otimes i-1} \otimes c \otimes I_m^{\otimes n-i-1}$$

for each $1 \leq i \leq n-1$. We have the following:

Proposition 3.12. *Suppose (V, c) is a (k, m) -generalized braided vector space. The assignment $\sigma_i \rightarrow c_i^{k, m}$ defines a group homomorphism for $n \geq 1$:*

$$\rho^c : \mathcal{B}_n \rightarrow \text{Aut}_{\mathbb{C}}(V^{\otimes k+m(n-2)}).$$

Proof. Eqn. (3.7) implies that $c_1^{k, m} c_2^{k, m} c_1^{k, m} = c_2^{k, m} c_1^{k, m} c_2^{k, m}$ and it follows that $c_i^{k, m}$ and $c_{i+1}^{k, m}$ satisfy relation (1.2). For relation (1.1) it is enough to check that $c_1^{k, m}$ commutes with $c_3^{k, m}, \dots, c_{n-1}^{k, m}$, which is eqn. (3.8). \square

Remark 3.13. A result analogous to Prop. 3.12 for (k, m) -gYB operators on objects in arbitrary monoidal categories \mathcal{C} can be proved with essentially no change except to eqn. (3.9).

Example 3.14. Set $J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$. In [RZWG] it is shown that the matrix

$R = \frac{1}{\sqrt{2}} \begin{pmatrix} I & J \\ -J & I \end{pmatrix}$ yields a unitary $(3, 2)$ -generalized braided vector space (\mathbb{C}^2, R) (where I represents the 4×4 identity matrix).

4. LOCALIZATIONS

In this section we give some variations on the notion of localization in somewhat more flexible settings. Although the categories we have in mind are typically highly structured (abelian, semisimple etc.) and confer significant structure on the associated braid representations, we discard as much of these restrictions as is possible.

4.1. Sequences of algebras under $\mathbb{C}\mathcal{B}$.

Definition 4.1. A \mathbb{C} -category (resp. a C^* -category) \mathcal{A} shall be called **diagonal** if:

- (1) $\text{Obj}(\mathcal{A}) = \mathbb{N}$ (we shall denote by $[n]$ the object corresponding to the natural n),
- (2) for all $n, m \in \mathbb{N}, m \neq n$, $\text{Hom}_{\mathcal{A}}([m], [n]) = 0$.

To specify a diagonal category \mathcal{A} it is enough to describe the algebras $\text{End}_{\mathcal{A}}([n])$ for each n . We shall denote by $\mathbb{C}\mathcal{B}$ the diagonal \mathbb{C} -linear $*$ -category where $\text{End}_{\mathbb{C}\mathcal{B}}([n]) = \mathbb{C}\mathcal{B}_n$ for all $n \in \mathbb{N}$. Observe that this is just the linearization of the category *Braid* that appears in [JS] (as a $*$ -category).

Definition 4.2. (1) Let \mathcal{A} and \mathcal{D} be diagonal categories. A morphism $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{D}$ from \mathcal{A} to \mathcal{D} is a \mathbb{C} -linear functor $F : \mathcal{A} \rightarrow \mathcal{D}$ such that $\mathcal{F}([n]) = [n]$ for all $n \in \mathbb{N}$.

- (2) A **diagonal category under \mathbb{CB}** is a pair $(\mathcal{A}, \rho_{\mathcal{A}})$ where \mathcal{A} is a diagonal category and $\rho_{\mathcal{A}} : \mathbb{CB} \rightarrow \mathcal{A}$ is a morphism of diagonal categories.

To specify a morphism $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{D}$ of diagonal categories one need only describe $\mathcal{F} : \text{End}_{\mathcal{A}}([n]) \rightarrow \text{End}_{\mathcal{D}}([n])$ for each n , so that a diagonal category under \mathbb{CB} is a family of \mathbb{C} -algebra representations $\rho_{\mathcal{A}} : \mathbb{CB}_n \rightarrow \text{End}_{\mathcal{A}}([n])$.

In [TW] tensor representations $F : \text{Tang} \rightarrow \text{Vec}_f$ of the tangle category Tang are defined as covariant tensor functors such that $F([n]) = V^{\otimes n}$ for some $V \in \text{Vec}_f$. Since \mathbb{CB} is a subcategory of (the \mathbb{C} -linearization of) Tang such a tensor representation gives rise to a diagonal category under \mathbb{CB} .

Notation 4.3. If $(\mathcal{A}, \rho_{\mathcal{A}})$ is a diagonal category under \mathbb{CB} , we shall denote by $(\underline{\mathcal{A}}, \rho_{\mathcal{A}})$ the diagonal category under \mathbb{CB} given by $\text{End}_{\underline{\mathcal{A}}}([n]) := \rho_{\mathcal{A}}(\mathbb{CB}_n) \subset \text{End}_{\mathcal{A}}([n])$, and we shall denote by $\underline{\mathcal{A}}_{+1}$, the diagonal category under \mathbb{CB} where $\text{Hom}_{\underline{\mathcal{A}}_{+1}}([n], [m]) = \text{Hom}_{\underline{\mathcal{A}}}([n+1], [m+1])$, for all $n \in \mathbb{N}$.

Definition 4.4. Let \mathcal{A} be a diagonal category. A **representation of \mathcal{A}** is an indexed family (ϑ_n, V_n) ($n \geq 1$) of representations $\vartheta_n : \text{End}_{\mathcal{A}}([n]) \rightarrow \text{End}_{\mathbb{C}}(V_n)$.

If (ϑ_n, V_n) is a representation of a diagonal category $(\mathcal{A}, \rho_{\mathcal{A}})$ under \mathbb{CB} then we obtain another diagonal category $(\vartheta(\mathcal{A}), \vartheta \circ \rho_{\mathcal{A}})$ under \mathbb{CB} by setting $\text{End}_{\vartheta(\mathcal{A})}([n]) := \text{End}_{\mathbb{C}}(V_n)$ and

$$\vartheta(\rho_{\mathcal{A}}(\mathbb{CB}_n)) := \vartheta_n \circ \rho_{\mathcal{A}}(\mathbb{CB}_n) \subset \text{End}_{\mathbb{C}}(V_n) = \text{End}_{\vartheta(\mathcal{A})}([n]).$$

The following is a (realization-independent) replacement of Definition 1.1:

Definition 4.5. A **sequence of algebras under \mathbb{CB}** is a triple $(\mathcal{A}, \rho_{\mathcal{A}}, \iota_{\mathcal{A}})$, where $(\mathcal{A}, \rho_{\mathcal{A}})$ is a diagonal \mathbb{C} -category under \mathbb{CB} and $\iota_{\mathcal{A}} : \underline{\mathcal{A}} \rightarrow \underline{\mathcal{A}}_{+1}$ is a faithful morphism such that for all $n \in \mathbb{N}$ the following diagram commutes:

$$\begin{array}{ccc} \mathbb{CB}_n & \xrightarrow{\quad} & \mathbb{C}\rho_{\mathcal{A}}(\mathcal{B}_n) \\ \downarrow \iota & & \downarrow \iota_{\mathcal{A}} \\ \mathbb{CB}_{n+1} & \xrightarrow{\quad} & \mathbb{C}\rho_{\mathcal{A}}(\mathcal{B}_{n+1}) \end{array}$$

A sequence of algebras under \mathbb{CB} is called **unitary** if \mathcal{A} is a C^* -category and the morphisms $\rho_{\mathcal{A}}$ and $\iota_{\mathcal{A}}$ are $*$ -functors.

When the maps $\rho_{\mathcal{A}}$ and $\iota_{\mathcal{A}}$ are clear from the context we will often just write \mathcal{A} for a sequence of algebras under \mathbb{CB} .

Remarks 4.6. (1) Clearly if $(\mathcal{A}, \rho_{\mathcal{A}}, \iota_{\mathcal{A}})$ is a sequence of algebras under \mathbb{CB} then so is $(\underline{\mathcal{A}}, \rho_{\mathcal{A}}, \iota_{\mathcal{A}})$.

- (2) Suppose $(\mathcal{A}, \rho_{\mathcal{A}}, \iota_{\mathcal{A}})$ is a sequence of algebras under \mathbb{CB} and (ϑ_n, V_n) is a *faithful* representation of $(\underline{\mathcal{A}}, \rho_{\mathcal{A}})$ so that $\vartheta_n : \text{End}_{\underline{\mathcal{A}}}([n]) \cong \text{End}_{\vartheta(\mathcal{A})}([n])$ is invertible for all n . Then $(\vartheta(\mathcal{A}), \vartheta \circ \rho_{\mathcal{A}}, \vartheta_{n+1} \circ \iota_{\mathcal{A}} \circ \vartheta_n^{-1})$ is a sequence of

algebras under \mathbb{CB} . Moreover, setting $\rho_n = \vartheta_n \circ \rho_A : \mathbb{CB}_n \rightarrow \text{End}_{\mathbb{C}}(V_n)$ we obtain a sequence of braid representations (ρ_n, V_n) (in the sense of Definition 1.1) with injective algebra homomorphisms

$$\tau_n = \vartheta_{n+1} \iota_A \vartheta_n^{-1} : \rho_n(\mathbb{CB}_n) \hookrightarrow \rho_{n+1}(\mathbb{CB}_{n+1})$$

satisfying $\tau_n \circ \rho_n = \rho_{n+1} \circ \iota$.

- (3) A sequence of braid representations (ρ_n, V_n) as in Definition 1.1 gives rise to a sequence of algebras under \mathbb{CB} as follows: Let $\mathcal{S} = \mathcal{S}(\rho_n, V_n)$ denote the diagonal category with $\text{End}_{\mathcal{S}}([n]) := \text{End}_{\mathbb{C}}(V_n)$ and define $\rho_{\mathcal{S}} = \rho_n : \mathbb{CB}_n \rightarrow \text{End}_{\mathcal{S}}([n])$. The injective algebra maps τ_n in Definition 1.1 give us

$$\iota_{\mathcal{S}} = \tau_n : \mathbb{C}\rho_n(\mathcal{B}_n) = \text{End}_{\mathcal{S}}([n]) \rightarrow \text{End}_{\mathcal{S}}([n+1]) = \mathbb{C}\rho_{n+1}(\mathcal{B}_{n+1})$$

so that $(\mathcal{S}, \rho_{\mathcal{S}}, \iota_{\mathcal{S}})$ is a sequence of algebras under \mathbb{CB} , which is unitary if and only if the \mathcal{B}_n -representation V_n is unitary for all n .

4.2. Key examples. Sequences of algebras under \mathbb{CB} arise naturally in a number of settings. We single out some important examples for later use.

Example 4.7. Let \mathcal{C} be a monoidal \mathbb{C} -linear category. Given a Yang-Baxter operator c on $V \in \mathcal{C}$, let $(YB_{(V,c)}, \rho^{(V,c)}, \iota)$ be the sequence of algebras under \mathbb{CB} defined as follows:

- (1) $\text{End}_{YB_{(V,c)}}([n]) = \text{End}_{\mathcal{C}}(V^{\otimes n})$,
- (2) $\iota : \text{End}_{YB_{(V,c)}}([n]) \rightarrow \text{End}_{YB_{(V,c)}}([n+1])$ is defined by $\iota(f) = f \otimes I_V$ and
- (3) $\rho_n^{(V,c)} : \mathbb{CB}_n \rightarrow \text{End}_{YB_{(V,c)}}([n])$ is defined by $\rho_n^{(V,c)}(\sigma_i) = c_i$ where c_i is as in eqn (3.2).

If H is a quasi-triangular (quasi-)Hopf algebra and V is any H -module then c is a Yang-Baxter operator on V in the monoidal category $\text{Rep}(H)$. Thus $YB_{(V,c)}$ has the structure of a sequence of algebras under \mathbb{CB} .

Example 4.8. If \mathcal{C} is a braided fusion category with simple objects $\mathcal{O}(\mathcal{C}) = \{\mathbf{1} = X_0, \dots, X_{k-1}\}$ and $X \in \mathcal{O}(\mathcal{C})$ then the braiding c is a Yang-Baxter operator on X . The semisimplicity of \mathcal{C} implies that the sequence of algebras $(YB_{(X,c)}, \rho^{(X,c)}, \iota)$ under \mathbb{CB} has a faithful representation: (ϑ_n, W_n^X) where $W_n^X := \bigoplus_i \text{Hom}_{\mathcal{C}}(X_i, X^{\otimes n})$ is the minimal faithful $\text{End}_{\mathcal{C}}(X^{\otimes n})$ -module. This gives rise to the sequence of \mathcal{B}_n -representations of interest in [RW].

Example 4.9. Naturally \mathbb{CB} itself has the structure of a sequence of algebra under \mathbb{CB} . One may construct more sequences of algebras under \mathbb{CB} as quotients of \mathbb{CB} provided certain compatibility constraints are satisfied. For example we can define a sequence of algebras $\mathcal{TL}_0(q, \ell)$ under \mathbb{CB} from the Temperley-Lieb algebras by setting $\text{End}_{\mathcal{TL}_0(q, \ell)}([n]) = TL_n(q)/\text{Ann}(\text{tr}_{\ell})$ where $q = e^{2\pi i/\ell}$ and $\text{Ann}(\text{tr}_{\ell})$ is the annihilator of the associated (Markov) trace form on $TL_n(q)$. The maps $\rho_{\mathcal{TL}_0(q, \ell)}$ and $\iota_{\mathcal{TL}_0(q, \ell)}$ come from the compatibility of the surjection $\mathbb{CB}_n \twoheadrightarrow TL_n(q)/\text{Ann}(\text{tr}_{\ell})$

with $\iota : \mathcal{B}_n \rightarrow \mathcal{B}_{n+1}$. The sequence of algebras $\mathcal{TL}_0(q, \ell)$ is equivalent to the sequence $(YB_{(X,c)}, \rho^{(X,c)}, \iota)$ obtained from the generating simple object X (analogous to the vector representation of \mathfrak{sl}_2) in the braided fusion category $\mathcal{C}(\mathfrak{sl}_2, \ell)$. Indeed, $\text{End}_{\mathcal{C}(\mathfrak{sl}_2, \ell)}(X^{\otimes n}) \cong TL_n(q)/\text{Ann}(\text{tr}_\ell)$.

4.3. \mathcal{C} -localization.

Definition 4.10. Let \mathcal{A} be a sequence of algebras under \mathbb{CB} and \mathcal{C} be a monoidal \mathbb{C} -category. A **\mathcal{C} -localization** of \mathcal{A} is a Yang-Baxter operator c on $V \in \mathcal{C}$ and a faithful morphism $\phi : \underline{\mathcal{A}} \rightarrow YB_{(V,c)}$ such that $\phi \circ \rho_{\mathcal{A}} = \rho^{(V,c)}$.

A **unitary \mathcal{C} -localization** of a unitary sequence of algebras \mathcal{A} under \mathbb{CB} is a \mathcal{C} -localization c on V in a C^* -tensor category \mathcal{C} , such that c is a unitary isomorphism and ϕ is a $*$ -functor.

The sequences of algebras under \mathbb{CB} constructed from Yang-Baxter operators in a category \mathcal{C} obviously are localizable over \mathcal{C} using the same Yang-Baxter operator. This localization shall be called the **trivial localization** and this kind of localizations are not relevant for us.

The following result gives some non-trivial localizations for sequences of algebras associated to Yang-Baxter operators.

Proposition 4.11. *Let \mathcal{C}, \mathcal{D} be monoidal \mathbb{C} -linear categories and $F : \mathcal{C} \rightarrow \mathcal{D}$ a faithful strict monoidal functor. Then for every Yang-Baxter operator (V, σ) the pair $(F(V), F(\sigma))$ is Yang-Baxter operator and defines a \mathcal{D} -localization of $YB_{(V,\sigma)}$.*

That Definition 4.10 is a generalization of Definition 1.2 is demonstrated in:

Proposition 4.12. *Let (ρ_n, V_n) be a sequence of \mathcal{B}_n -representations (in the sense of Definition 1.1) and $(\mathcal{S}, \rho_{\mathcal{S}}, \iota_{\mathcal{S}})$ the associated sequence of algebras under \mathbb{CB} as in Remarks 4.6. Then (ρ_n, V_n) is localizable in sense of Definition 1.2 if and only if $(\mathcal{S}, \rho_{\mathcal{S}}, \iota_{\mathcal{S}})$ is Vec_f -localizable.*

Proof. First suppose that the sequence of \mathcal{B}_n -representations (ρ_n, V_n) has localization (V, c) as in Definition 1.2. Then there are injective maps

$$\phi_n : \text{End}_{\underline{\mathcal{S}}}([n]) = \mathbb{C}\rho_n(\mathcal{B}_n) \rightarrow \text{End}(V^{\otimes n}) = \text{End}_{YB_{(V,c)}}([n])$$

such that $\phi_n \circ \rho_{\mathcal{S}} = \rho_n^{(V,c)}$ for all $n \in \mathbb{N}$. Thus ϕ_n defines a faithful morphism $\phi : \underline{\mathcal{S}} \rightarrow YB_{(V,c)}$ such that $\phi \circ \rho_{\mathcal{S}} = \rho^{(V,c)}$, i.e., a Vec_f -localization of $(\mathcal{S}, \rho_{\mathcal{S}}, \iota_{\mathcal{S}})$.

Conversely, a Vec_f -localization of $(\mathcal{S}, \rho_{\mathcal{S}}, \iota_{\mathcal{S}})$ is a braided vector space (V', c') and a faithful morphism $\phi : \underline{\mathcal{S}} \rightarrow YB_{(V',c')}$ such that $\phi \circ \rho_{\mathcal{S}} = \rho^{(V',c')}$, so $\phi_n : \text{End}_{\underline{\mathcal{S}}}([n]) = \mathbb{C}\rho_n(\mathcal{B}_n) \rightarrow \text{End}_{YB_{(V',c')}}([n]) = \text{End}(V'^{\otimes n})$ define a localization of (ρ_n, V_n) in the sense of Definition 1.2. \square

This motivates the following short-hand notation:

Definition 4.13. A Vec_f -localization of \mathcal{A} will be called a **localization** and a (unitary) Hilb_f -localization of \mathcal{A} will be called a **unitary localization**.

If (V, c) is a localization of \mathcal{A} we will refer to $\dim(V)$ as the **dimension** of the localization.

Corollary 4.14. *Let $(\mathcal{A}, \rho_{\mathcal{A}}, \iota_{\mathcal{A}})$ be a localizable sequence of algebras under \mathbb{CB} . Then for any faithful representation (ϑ_n, V_n) of \mathcal{A} the sequence of \mathcal{B}_n -representations $(\vartheta_n \circ \rho_{\mathcal{A}}, V_n)$ is localizable in the sense of Definition 1.2.*

□

We illustrate this result with the following example.

Example 4.15. Consider the sequence $(YB_{(X,c)}, \rho^{(X,c)}, \iota)$ constructed from an object in a braided fusion category \mathcal{C} as in Example 4.8. The faithful representation (ϑ_n, W_n^X) of $YB_{(X,c)}$ yields a sequence of \mathcal{B}_n -representations $(\vartheta_n \circ \rho^{(X,c)}, W_n^X)$. If $YB_{(X,c)}$ has localization (V, R) with faithful morphism $\phi : YB_{(X,c)} \rightarrow \text{End}_{\mathcal{C}}(V^{\otimes n})$ satisfying $\phi \circ \rho^{(X,c)} = \rho^{(V,R)}$ then defining $\Phi_n = \phi \circ (\vartheta_n \mid_{\rho^{(X,c)}(\mathbb{CB}_n)})^{-1}$ gives us the required algebra injections $(\vartheta_n \circ \rho^{(X,c)})(\mathbb{CB}_n) \xrightarrow{\Phi_n} \text{End}_{\mathcal{C}}(V^{\otimes n})$ so that $(\vartheta_n \circ \rho^{(X,c)}, W_n^X)$ is localizable.

4.4. Quasi-localization.

Definition 4.16. Let \mathcal{A} be a sequence of algebras under \mathbb{CB} . A **quasi-localization** of \mathcal{A} is a (V, a, c) -localization where (V, a, c) is a quasi-braided vector space.

Again, $\dim(V)$ will be the **dimension** of the quasi-localization (V, a, c) .

Proposition 4.17. *Let H be a quasi-triangular (resp. quasi-Kac algebra) quasi-Hopf algebra and V an (resp. unitary) H -module. Then the sequence of (resp. unitary) algebras $YB_{(V,c)}$ under \mathbb{CB} has a (resp. unitary) quasi-localization of dimension $\dim(V)$. Moreover, if H is a (resp. Kac algebra) Hopf algebra, the sequence of (resp. unitary) algebras $YB_{(V,c)}$ under \mathbb{CB} has a (resp. unitary) localization of dimension $\dim(V)$.*

Proof. Let (V, a, c) be the quasi-braided vector space defined in Example 3.7. Then the monoidal category $\langle V \rangle \subset \text{Rep}(H)$ is a full tensor subcategory of (V, a, c) so by Proposition 4.11 (V, a, c) is quasi-localization of $YB_{(V,c)}$. If H is a quasitriangular quasi-Kac algebra (V, a, c) is a unitary quasi-braided vector space and (V, a, c) is a C^* -tensor category, so the same argument shows that (V, a, c) is a unitary quasi-localization of $YB_{(V,c)}$.

Finally, if H is a Hopf algebra (resp. Kac algebra) we can choose a trivial, so (V, c) is a localization (resp. unitary localization). □

Remark 4.18. (1) Observe that if \mathcal{C} is an integral braided fusion category then there is a quasi-Hopf algebra H such that $\mathcal{C} \cong \text{Rep}(H)$ by [ENO1, Theorem 8.33]. In general it is difficult to determine if \mathcal{C} actually has a fiber functor, i.e. H can be chosen to be a (coassociative) Hopf algebra. Prop. 4.17 shows that if the braid group representation associated with

$X \in \mathcal{C}$ does not have a localization (V, c) with $\dim(V) = \text{FPdim}(X)$ then no fiber functor can exist.

- (2) Note that Prop. 4.17 holds for *topological* quasi-triangular quasi-Hopf algebras (see [Ks]): we may allow the universal R -matrix to reside in a completion of $H \otimes H$ as long as it gives rise to a quasi-braided vector space (V, a, c) .

In analogy with Prop. 4.12 we use the following:

Notation 4.19. Let (ρ_n, V_n) be a sequence of braid representations and \mathcal{S} the associated sequence of algebras under \mathbb{CB} . A quasi-localization (V, a, c) of \mathcal{S} will be also called a quasi-localization of (ρ_n, V_n) .

4.5. Weak localization. In analogy with weak Hopf algebras we make the following:

Definition 4.20. Let \mathcal{A} be a sequence of algebras under \mathbb{CB} . A **weak localization** of \mathcal{A} is a $\text{Bimod}(R)$ -localization of \mathcal{A} where R is a finite dimensional semisimple \mathbb{C} -algebra.

Any (multi-)fusion category is equivalent to the representation category of a weak Hopf algebra ([ENO1, Corollary 2.22]). We have the related:

Proposition 4.21. *Let \mathcal{C} be a fusion category and c a Yang-Baxter operator on $V \in \mathcal{C}$. The sequence of algebras $YB_{(V,c)}$ has a weak-localization.*

Proof. By [O1] every fusion category admits a faithful exact monoidal functor from \mathcal{C} to $\text{Bimod}(R)$ for some R , so the proposition follows from Proposition 4.11. \square

- Remarks 4.22.** (1) A weak localization with R a simple algebra, defines a localization in Vec_f . In fact, since R is simple, $\text{Bimod}(R)$ is monoidally equivalent to Vec_f .
 (2) If \mathcal{C} is a fusion category, by [O1] there is a bijective correspondence between structures of \mathcal{C} -module categories and faithful exact monoidal functors from \mathcal{C} to $\text{Bimod}(R)$ for R a semisimple algebra.

4.6. Generalized localization.

Example 4.23. Given a (k, m) -generalized braided vector space (V, c) , we define $(gYB_{(V,c)}, \rho^{(V,c)}, \iota)$ to be the sequence of algebras under \mathbb{CB} defined as follows:

- (1) $\text{End}_{gYB_{(V,c)}}([n]) = \text{End}(V^{\otimes k+m(n-2)})$
- (2) $\iota : \text{End}_{gYB_{(V,c)}}([n]) \rightarrow \text{End}_{gYB_{(V,c)}}([n+1])$ is defined by $f \mapsto f \otimes \text{I}_V^{\otimes m}$ and
- (3) $\rho_n^{(V,c)} : \mathbb{CB}_n \rightarrow \text{End}_{gYB_{(V,c)}}([n])$ is defined by $\sigma_i \mapsto c_i^{k,m}$ where $c_i^{k,m}$ is given by eqn. (3.9).

Definition 4.24. Let \mathcal{A} be a sequence of algebras under \mathbb{CB} . A **(k, m) -localization of \mathcal{A}** is a (k, m) -generalized braided vector space (V, c) and a faithful morphism $\phi : \underline{\mathcal{A}} \rightarrow gYB_{(V,c)}$ such that $\phi \circ \rho_{\mathcal{A}} = \rho^{(V,c)}$.

A **unitary (k, m) -localization** of a unitary sequence of algebras under \mathbb{CB} is a (k, m) -localization (V, c) over the C^* -tensor category Hilb_f , such that c is a unitary isomorphism and ϕ is a $*$ -functor.

Remark 4.25. Notice that one may define (k, m) -localizations over *any* monoidal category \mathcal{C} by modifying Example 4.23. We do not do so as we are ultimately interested in matrix representations of the braid groups.

As in the case of quasi-localizations we use the following:

Notation 4.26. Let (ρ_n, V_n) be a sequence of braid representations and \mathcal{S} the associated sequence of algebras under \mathbb{CB} . A (k, m) -localization (V, c) of \mathcal{S} will be also called a (k, m) -localization of (ρ_n, V_n) .

5. UNITARITY OF WEAKLY GROUP-THEORETICAL FUSION CATEGORIES

5.1. The dual of a \mathcal{C} -module $*$ -category.

Definition 5.1. A **weak C^* -Hopf algebra** (resp. a **quasi- C^* -Hopf algebra**) is a weak Hopf algebra $(H, m, \Delta, \varepsilon)$ (resp. a quasi-Hopf algebra $(H, m, \Delta, \varepsilon, \Phi, S)$), such that H is a finite dimensional C^* -algebra and Δ is a $*$ -homomorphism (resp. Δ is a $*$ -homomorphism and $\Phi^* = \Phi^{-1}$).

Remark 5.2. The uniqueness of the unit, counit and the antipode for weak Hopf algebras (see [BNSz, Proposition 2.10]) imply that

$$1^* = 1, \quad \varepsilon(x^*) = \overline{\varepsilon(x)}, \quad (S \circ *)^2 = I_H.$$

The dual \widehat{H} of a weak C^* -Hopf algebra is again a C^* -algebra with $*$ -operation (see [BNSz, Theorem 4.5])

$$\langle \phi^*, x \rangle = \langle \phi, \overline{S(x)} \rangle, \quad \text{for all } \phi \in \widehat{H}, x \in H.$$

A **$*$ -representation** of a weak C^* -Hopf algebra H is a finite dimensional Hilbert space $(V, \langle \cdot, \cdot \rangle_V)$ carrying a left action of H , such that $\langle u, x \cdot v \rangle_V = \langle x^* \cdot u, v \rangle_V$ for all $u, v \in V$ and $x \in H$. The morphisms from $(V, \langle \cdot, \cdot \rangle_V)$ to $(W, \langle \cdot, \cdot \rangle_W)$ are defined to be the H -module morphisms from V to W . The category so obtained will be denoted by $\mathcal{U}\text{-Rep}(H)$, and it is a unitary (multi)-fusion category, see [BSz, Section 3].

Theorem 5.3. *Let \mathcal{C} be a unitary fusion category and \mathcal{M} be a \mathcal{C} -module $*$ -category with \mathcal{M} indecomposable. Then the monoidal category $\text{End}_{\mathcal{C}}^*(\mathcal{M})$ is a unitary fusion category monoidal equivalent to $\text{End}_{\mathcal{C}}(\mathcal{M})$.*

Proof. Let R be a finite dimensional C^* -algebra such that $\mathcal{U}\text{-Rep}(R)$ is unitarily equivalent to \mathcal{M} . By [O1, Proposition 3] the \mathcal{C} -module structure on \mathcal{M} defines an R -fiber functor $F : \mathcal{C} \rightarrow \text{Bimod}(R)$, so by [O1, Theorem 4] there is a canonical weak Hopf algebra H such that $\text{Rep}(H)$ is monoidally equivalent to \mathcal{C} and $\text{Rep}(\widehat{H})$

is monoidally equivalent to $\text{End}_{\mathcal{C}}(\mathcal{M})$. Since \mathcal{M} is a \mathcal{C} -module $*$ -category the R -fiber functor is a $*$ -monoidal functor, and H is a weak C^* -Hopf algebra. The dual weak Hopf algebra \hat{H} is again a weak C^* -Hopf algebra, so $\text{End}_{\mathcal{C}}(\mathcal{M})$ is a unitary fusion category. Finally, in order to prove that $\text{End}_{\mathcal{C}}^*(\mathcal{M})$ is $*$ -monoidally equivalent to $\text{End}_{\mathcal{C}}(\mathcal{M})$, we can use again [O1, Theorem 4] and that every \hat{H} -module is isomorphic to a unitary \hat{H} -module. \square

5.2. Tensor products of module categories. In this section we shall define the tensor product of module $*$ -categories over unitary fusion categories, following [ENO3]. We shall denote the tensor product of \mathcal{C} -module categories defined in *loc. cit.* by $\boxtimes_{\mathcal{C}}$.

Definition 5.4. Let \mathcal{M}_1 and \mathcal{M}_2 be $*$ -categories. The exterior tensor product $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ is the $*$ -category with following objects and morphisms:

$$\begin{aligned} \text{Obj}(\mathcal{M}_1 \boxtimes \mathcal{M}_2) &= \left\{ \bigoplus_{i \in \mathcal{I}} X_i \boxtimes Y_i : X_i \in \text{Obj}(\mathcal{M}_1), Y_i \in \text{Obj}(\mathcal{M}_2), |\mathcal{I}| < \infty \right\}, \\ \text{Hom}_{\mathcal{M}_1 \boxtimes \mathcal{M}_2} \left(\bigoplus_{i \in \mathcal{I}} X_i \boxtimes Y_i, \bigoplus_{i \in \mathcal{I}'} X'_i \boxtimes Y'_i \right) &= \bigoplus_{i, j \in \mathcal{I}} \text{Hom}_{\mathcal{M}_1}(X_i, X'_j) \otimes_{\mathbb{C}} \text{Hom}_{\mathcal{M}_2}(Y_i, Y'_j), \end{aligned}$$

and $*$ -structure $(f \boxtimes g)^* = f^* \boxtimes g^*$.

Let \mathcal{C}, \mathcal{D} be unitary fusion categories, so that the $*$ -category $\mathcal{C} \boxtimes \mathcal{D}$ has an obvious $*$ -fusion category structure. By definition, a $(\mathcal{C}, \mathcal{D})$ -bimodule $*$ -category is a module category over the unitary fusion category $\mathcal{C} \boxtimes \mathcal{D}^{\text{rev}}$, where \mathcal{D}^{rev} is \mathcal{D} with reversed tensor product.

For the following two definitions let \mathcal{A} be a $*$ -category and \mathcal{M}, \mathcal{N} left and right (strict) \mathcal{C} -module $*$ -categories, respectively.

Definition 5.5. [ENO3, Definition 3.1] Let $F : \mathcal{M} \boxtimes \mathcal{N} \rightarrow \mathcal{A}$ be an exact $*$ -functor. We say that F is **\mathcal{C} -balanced** if there is a natural family of unitary isomorphisms

$$b_{M, X, N} : F(M \otimes X \boxtimes N) \rightarrow F(M \boxtimes X \otimes N),$$

such that

$$b_{M, X \otimes Y, N} = b_{M, X, Y \otimes N} \circ b_{M \otimes X, Y, N},$$

for all $M \in \mathcal{M}, N \in \mathcal{N}, X, Y \in \mathcal{C}$.

Definition 5.6. [ENO3, Definition 3.3] A tensor product of a right \mathcal{C} -module $*$ -category \mathcal{M} and a left \mathcal{C} -module $*$ -category \mathcal{N} is a $*$ -category $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ together with a \mathcal{C} -balanced $*$ -functor

$$B_{\mathcal{M}, \mathcal{N}} : \mathcal{M} \boxtimes \mathcal{N} \rightarrow \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$$

inducing, for every $*$ -category \mathcal{A} , an equivalence between the category of \mathcal{C} -balanced $*$ -functors from $\mathcal{M} \boxtimes \mathcal{N}$ to \mathcal{A} and the category of $*$ -functors from $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ to \mathcal{A} :

$$\text{Hom}_{bal}^*(\mathcal{M} \boxtimes \mathcal{N}, \mathcal{A}) \cong \text{Hom}^*(\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}, \mathcal{A}).$$

Remark 5.7. (1) The existence of the tensor product for module categories over $*$ -fusion categories can be proved using the same ideas in [ENO3]. semisimple category, see [ENO3].

Given a right \mathcal{C} -module $*$ -functor $F : \mathcal{M} \rightarrow \mathcal{M}'$ and a left \mathcal{C} -module $*$ -functor $G : \mathcal{N} \rightarrow \mathcal{N}'$ note that $B_{\mathcal{M}', \mathcal{N}'}(F \boxtimes G) : \mathcal{M} \boxtimes \mathcal{N} \rightarrow \mathcal{M}' \boxtimes \mathcal{N}'$ is a \mathcal{C} -balanced $*$ -functor. Thus the universality of B implies the existence of a unique right $*$ -functor $F \boxtimes_{\mathcal{C}} G := \overline{B_{\mathcal{M}', \mathcal{N}'}(F \boxtimes G)}$ making the diagram

$$\begin{array}{ccc} \mathcal{M} \boxtimes \mathcal{N} & \xrightarrow{F \boxtimes G} & \mathcal{M}' \boxtimes \mathcal{N}' \\ B_{\mathcal{M}, \mathcal{N}} \downarrow & & \downarrow B_{\mathcal{M}', \mathcal{N}'} \\ \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} & \xrightarrow{F \boxtimes_{\mathcal{C}} G} & \mathcal{M}' \boxtimes_{\mathcal{C}} \mathcal{N}' \end{array}$$

commutative. The bihomomorphism $\boxtimes_{\mathcal{C}}$ is functorial in module $*$ -functors, *i.e.*, $(F' \boxtimes_{\mathcal{C}} E')(F \boxtimes_{\mathcal{C}} E) = F' F \boxtimes_{\mathcal{C}} E' E$.

Remarks 5.8. (1) If \mathcal{M} is a $(\mathcal{C}, \mathcal{E})$ -bimodule $*$ -category and \mathcal{N} is an $(\mathcal{E}, \mathcal{D})$ -bimodule $*$ -category, then $\mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N}$ is a $(\mathcal{C}, \mathcal{D})$ -bimodule $*$ -category and $B_{\mathcal{M}, \mathcal{N}}$ is a $(\mathcal{C}, \mathcal{D})$ -bimodule $*$ -functor, see [Gr, Proposition 3.13].
(2) Let \mathcal{M} be a $(\mathcal{C}, \mathcal{D})$ -bimodule $*$ -category. The \mathcal{C} -module action on \mathcal{M} defines a \mathcal{C} -balanced $*$ -functor. Let $r_{\mathcal{M}} : \mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{M} \rightarrow \mathcal{M}$ denote the unique $*$ -functor factoring through $B_{\mathcal{C}, \mathcal{M}}$. As in [Gr, Proposition 3.15] we can prove that $r_{\mathcal{M}}$ is a $(\mathcal{C}, \mathcal{D})$ -module $*$ -category equivalence.
(3) Let \mathcal{M} be a right \mathcal{C} -module $*$ -category, \mathcal{N} a $(\mathcal{C}, \mathcal{D})$ -bimodule $*$ -category, and \mathcal{K} a left \mathcal{D} -module $*$ -category. Then as in [Gr, Proposition 3.15], there is a canonical equivalence $(\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}) \boxtimes_{\mathcal{D}} \mathcal{K} \cong \mathcal{M} \boxtimes_{\mathcal{C}} (\mathcal{N} \boxtimes_{\mathcal{D}} \mathcal{K})$ of bimodule $*$ -categories. Hence the notation $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \boxtimes_{\mathcal{D}} \mathcal{K}$ will yield no ambiguity.

5.3. Crossed product tensor categories. We briefly recall group actions on tensor categories. For more details the reader is referred to [DGNO].

Let \mathcal{C} be a tensor category and let $\underline{\text{Aut}}_{\otimes}(\mathcal{C})$ be the monoidal category of monoidal auto-equivalences of \mathcal{C} , arrows are tensor natural isomorphisms and tensor product the composition of monoidal functors.

For any group G we shall denote by \underline{G} the monoidal category where objects are elements of G and tensor product is given by the product of G . An action of the group G on \mathcal{C} , is a monoidal functor $F : \underline{G} \rightarrow \underline{\text{Aut}}_{\otimes}(\mathcal{C})$. In other words, for any $\sigma \in G$ there is a monoidal functor $(F_{\sigma}, \zeta_{\sigma}) : \mathcal{C} \rightarrow \mathcal{C}$, and for any $\sigma, \tau \in G$, there are natural monoidal isomorphisms $\gamma_{\sigma, \tau} : F_{\sigma} \circ F_{\tau} \rightarrow F_{\sigma\tau}$.

Given an action $F : \underline{G} \rightarrow \underline{\text{Aut}}_{\otimes}(\mathcal{C})$ of G on \mathcal{C} , the G -crossed product tensor category, denoted by $\mathcal{C} \rtimes G$, is defined as follows. As an abelian category $\mathcal{C} \rtimes G =$

$\bigoplus_{\sigma \in G} \mathcal{C}_\sigma$, where $\mathcal{C}_\sigma = \mathcal{C}$ as an abelian category, the tensor product is

$$[X, \sigma] \otimes [Y, \tau] := [X \otimes F_\sigma(Y), \sigma\tau], \quad X, Y \in \mathcal{C}, \quad \sigma, \tau \in G,$$

and the unit object is $[1, e]$. See [Tam] or [Ga3] for the associativity constraint and a proof of the pentagon identity.

Lemma 5.9. *Let \mathcal{C} be a unitary fusion category and G a finite group acting on \mathcal{C} . Then $\mathcal{C} \rtimes G$ is a unitary fusion category if and only if the G -action on \mathcal{C} is unitary, i.e., F_σ are monoidal $*$ -functors and $\gamma_{\sigma, \tau}$ are unitary natural isomorphisms for all $\sigma, \tau \in G$. If $\mathcal{C} \rtimes G$ is a unitary fusion category \mathcal{C} is a $\mathcal{C} \rtimes G$ -module $*$ -category.*

Proof. Straightforward. \square

5.4. Clifford theory for G -graded fusion categories. Let G be a group and \mathcal{C} be a tensor category. We shall say that \mathcal{C} is G -graded if there is a decomposition

$$\mathcal{C} = \bigoplus_{\sigma \in G} \mathcal{C}_\sigma$$

of \mathcal{C} into a direct sum of full abelian subcategories, such that for all $\sigma, \tau \in G$, the bifunctor \otimes maps $\mathcal{C}_\sigma \times \mathcal{C}_\tau$ to $\mathcal{C}_{\sigma\tau}$. Given a G -graded tensor category \mathcal{C} , and a subgroup $H \subset G$, we shall denote by \mathcal{C}_H the tensor subcategory $\bigoplus_{h \in H} \mathcal{C}_h$.

Definition 5.10. Let \mathcal{C} be a G -graded fusion category. If $(\mathcal{M}, \overline{\otimes})$ is a \mathcal{C}_e -module category, then a **\mathcal{C} -extension of \mathcal{M}** is a \mathcal{C} -module category (\mathcal{M}, \odot) such that $(\mathcal{M}, \overline{\otimes})$ is obtained by restriction to \mathcal{C}_e .

Proposition 5.11. *Let \mathcal{C} be a unitary fusion category graded by a group G and $(\mathcal{M}, \overline{\otimes})$ an indecomposable \mathcal{C}_e -module $*$ -category. Then each \mathcal{C} -extension (\mathcal{M}, \odot) is a \mathcal{C} -module $*$ -category.*

Proof. Let (\mathcal{M}, \odot) be a \mathcal{C} -extension of $(\mathcal{M}, \overline{\otimes})$. If $\overline{\mathcal{M}} = \mathcal{C} \overline{\boxtimes}_{\mathcal{C}_e} \mathcal{M}$, then by [Ga2, Theorem 1.3] $\text{End}_{\mathcal{C}}^*(\overline{\mathcal{M}})$ is a unitary fusion category $*$ -monoidally equivalent to a G -semidirect product unitary fusion category $\text{End}_{\mathcal{C}_e}^*(\mathcal{M}) \rtimes G$ and the \mathcal{C} -extension (\mathcal{M}, \odot) is completely determined by the $\text{End}_{\mathcal{C}_e}^*(\mathcal{M}) \rtimes G$ -module category $\text{End}_{\mathcal{C}_e}^*(\mathcal{M})$. By Lemma 5.9 $\text{End}_{\mathcal{C}_e}^*(\mathcal{M})$ is a $\text{End}_{\mathcal{C}_e}^*(\mathcal{M}) \rtimes G$ -module $*$ -category, thus the \mathcal{C} -extension is a \mathcal{C} -module $*$ -category. \square

The following is a simplified version of Clifford theorem for unitary fusion category, see [Ga2].

Theorem 5.12. *Let \mathcal{C} be a G -graded $*$ -fusion category, \mathcal{M} an indecomposable \mathcal{C} -module $*$ -category and \mathcal{N} an indecomposable \mathcal{C}_e -submodule $*$ -subcategory of \mathcal{M} . Then there is a subgroup $S \subset G$ and a \mathcal{C}_S -extension (\mathcal{N}, \odot) of \mathcal{N} , such that $\mathcal{M} \cong \mathcal{C} \overline{\boxtimes}_{\mathcal{C}_S} \mathcal{N}$ as \mathcal{C} -module $*$ -categories.*

Proof. It follows from [Ga2, Corollary 4.4] and Proposition 5.11. \square

5.5. Completely unitary fusion categories.

Definition 5.13. Let \mathcal{C} be a fusion category. We shall say that \mathcal{C} is **completely unitary** if the following properties are satisfied:

- (1) \mathcal{C} is monoidally equivalent to a unique (up to $*$ -monoidal equivalences) unitary fusion category (we shall denote this unitary fusion category again by \mathcal{C}).
- (2) Every \mathcal{C} -module category is equivalent to a unique (up to \mathcal{C} -module $*$ -functor equivalences) \mathcal{C} -module $*$ -category.
- (3) Every \mathcal{C} -module functor equivalence between \mathcal{C} -module $*$ -categories is equivalent to a unique (up to unitary \mathcal{C} -module natural isomorphisms) \mathcal{C} -module $*$ -functor equivalence.

Remark 5.14. Let $U(1) = \{z \in \mathbb{C} : |z| = 1\}$ and G be a finite group. By the universal coefficient theorem [Rot, Theorem 10.22] $H^n(G, U(1)) = H^n(G, \mathbb{C}^*)$ for all $n > 0$, *i.e.*, every n -cocycle with coefficients on \mathbb{C}^* is equivalent to a some n -cocycle with coefficients on $U(1)$.

Proposition 5.15. *Every pointed fusion category is a completely unitary fusion category.*

Proof. It follows from Remark 5.14 and the classification of module categories over pointed fusion categories, [O2]. \square

In [ENO3] they show that a graded fusion category $\mathcal{C} = \bigoplus_{\sigma \in G} \mathcal{C}_\sigma$ determines and it is determined by the following data:

- (1) a fusion category \mathcal{C}_e , a collection of invertible \mathcal{C}_e -bimodule categories $\mathcal{C}_\sigma, \sigma \in G$,
- (2) a collection of \mathcal{C}_e -bimodule isomorphisms $M_{\sigma,\tau} : \mathcal{C}_\sigma \boxtimes_{\mathcal{C}_e} \mathcal{C}_\tau \rightarrow \mathcal{C}_{\sigma\tau}$,
- (3) natural isomorphisms of \mathcal{C}_e -bimodule functors

$$\alpha_{\sigma,\tau,\rho} : M_{\sigma,\tau\rho}(\text{Id}_{\mathcal{C}_\sigma} \boxtimes_{\mathcal{C}_e} M_{\tau,\rho}) \rightarrow M_{\sigma\tau,\rho}(M_{\sigma,\tau} \boxtimes_{\mathcal{C}_e} \text{Id}_{\mathcal{C}_\rho})$$

satisfying the identity

$$(5.1) \quad \begin{aligned} & M_{\sigma,\tau\rho k}(\text{Id}_\sigma \boxtimes_{\mathcal{C}_e} \alpha_{\tau,\rho,k}) \circ \alpha_{\sigma,\tau\rho,k}(\text{Id}_{\mathcal{C}_\sigma} \boxtimes_{\mathcal{C}_e} M_{\tau,\rho} \boxtimes_{\mathcal{C}_e} \text{Id}_{\mathcal{C}_k}) \\ &= \alpha_{\sigma,\tau,\rho k}(\text{Id}_{\mathcal{C}_\sigma} \boxtimes_{\mathcal{C}_e} \text{Id}_{\mathcal{C}_\tau} \boxtimes_{\mathcal{C}_e} M_{\rho,k}) \circ \alpha_{\sigma\tau,\rho,k}(M_{\sigma,\tau} \boxtimes_{\mathcal{C}_e} \text{Id}_{\mathcal{C}_\rho} \boxtimes_{\mathcal{C}_e} \text{Id}_{\mathcal{C}_k}), \end{aligned}$$

for all $\sigma, \tau, \rho, k \in G$, where we use the notation Id for the identity functor, and Id for the identity morphism.

Remark 5.16. If \mathcal{C} is a G -graded fusion category where \mathcal{C}_e is a unitary fusion category, then \mathcal{C} is a unitary fusion category with \mathcal{C}_e as unitary fusion subcategory if and only if \mathcal{C}_σ are \mathcal{C}_e -bimodule $*$ -category, $M_{\sigma,\tau}$ are \mathcal{C}_e -bimodules $*$ -functors, and $\alpha_{\sigma,\tau,\rho}$ are unitary isomorphism.

Theorem 5.17. *If \mathcal{C} is a G -graded fusion category such that \mathcal{C}_e and $\mathcal{C}_e \boxtimes \mathcal{C}_e^{rev}$ are completely unitary then \mathcal{C} and $\mathcal{C} \boxtimes \mathcal{C}^{rev}$ are completely unitary.*

Proof. First we shall show that \mathcal{C} is monoidally equivalent to a unique unitary fusion category. Since $\mathcal{C}_e \boxtimes \mathcal{C}_e^{rev}$ is completely unitary for each $\sigma \in G$, the \mathcal{C}_e -bimodule category \mathcal{C}_σ is equivalent to a unique \mathcal{C}_e -bimodule $*$ -category $\overline{\mathcal{C}}_\sigma$. The bifunctor $\otimes : \mathcal{C}_\sigma \times \mathcal{C}_\tau \rightarrow \mathcal{C}_{\sigma\tau}$ and the complete unitarity of $\mathcal{C}_e \boxtimes \mathcal{C}_e^{rev}$ define for each pair $\sigma, \tau \in G$ a unique \mathcal{C}_e -bimodule $*$ -functor $M_{\sigma,\tau} : \overline{\mathcal{C}}_\sigma \boxtimes_{\mathcal{C}_e} \overline{\mathcal{C}}_\tau \rightarrow \overline{\mathcal{C}}_{\sigma\tau}$, such that

$$M_{f,gh}(Id_{\mathcal{C}_f} \boxtimes_{\mathcal{C}_e} M_{g,h}) \cong M_{f,g,h}(M_{f,g} \boxtimes_{\mathcal{C}_e} Id_{\mathcal{C}_h}),$$

as \mathcal{C}_e -module functors. Now, using the polar decomposition (see Remark 2.4) and the associativity constraint of \mathcal{C} , there are unitary isomorphisms of \mathcal{C}_e -module $*$ -functors

$$\alpha_{\sigma,\tau,\rho} : M_{f,gh}(Id_{\mathcal{C}_f} \boxtimes_{\mathcal{C}_e} M_{g,h}) \rightarrow M_{f,g,h}(M_{f,g} \boxtimes_{\mathcal{C}_e} Id_{\mathcal{C}_h}),$$

for all $\sigma, \tau, \rho \in G$, such that the equation (5.1) holds. The new G -graded fusion category is equivalent to \mathcal{C} and it is a unitary fusion category.

Thus we may assume that \mathcal{C} is a unitary fusion category. Let \mathcal{M} be an indecomposable \mathcal{C} -module category, then by the complete unitarity of \mathcal{C}_e and Theorem 5.12, \mathcal{M} is equivalent to a \mathcal{C} -module $*$ -category. Moreover, if \mathcal{M} and \mathcal{N} are \mathcal{C} -module $*$ -categories equivalent as \mathcal{C} -module categories, by [Ga2, Proposition 4.6], Remark 5.14 and [Ga2, Theorem 1.3], \mathcal{M} and \mathcal{N} are equivalent as \mathcal{C} -module $*$ -categories and every \mathcal{C} -module equivalence is equivalent to a \mathcal{C} -module $*$ -functor equivalence.

Finally, note that $\mathcal{C} \boxtimes \mathcal{C}^{rev}$ is a $G \times G^{op}$ -graded fusion category where $(\mathcal{C} \boxtimes \mathcal{C}^{rev})_{(e,e)} = \mathcal{C}_e \boxtimes \mathcal{C}_e^{rev}$. Thus by the second part of this proof $\mathcal{C} \boxtimes \mathcal{C}^{rev}$ is completely unitary. \square

5.5.1. Weakly group-theoretical fusion categories are completely unitary. Let \mathcal{C} be an arbitrary fusion category. The adjoint category \mathcal{C}_{ad} is the smallest fusion subcategory of \mathcal{C} containing all objects $X \otimes X^*$, where $X \in \mathcal{C}$ is simple. There exists a unique faithful grading of \mathcal{C} for which $\mathcal{C}_e = \mathcal{C}_{ad}$ (see [GeNik]). It is called the universal grading of \mathcal{C} . The corresponding group is called the universal grading group of \mathcal{C} , and denoted by $U(\mathcal{C})$. All faithful gradings of \mathcal{C} are induced by the universal grading, in the sense that for any faithful grading $U(\mathcal{C})$ canonically projects onto the grading group G , and \mathcal{C}_e contains \mathcal{C}_{ad} .

Let \mathcal{C} be a fusion category. Let $\mathcal{C}^{(0)} = \mathcal{C}$, $\mathcal{C}^{(1)} = \mathcal{C}_{ad}$ and $\mathcal{C}^{(n)} = (\mathcal{C}^{(n-1)})_{ad}$ for every integer $n \geq 1$. The non-increasing sequence of fusion subcategories of \mathcal{C}

$$\mathcal{C} = \mathcal{C}^{(0)} \supseteq \mathcal{C}^{(1)} \supseteq \dots \supseteq \mathcal{C}^{(n)} \supseteq \dots$$

is called the upper central series of \mathcal{C} .

Definition 5.18. [GeNik] A fusion category \mathcal{C} is called **nilpotent** if its upper central series converges to Vec_f ; i.e., $\mathcal{C}^{(n)} = \text{Vec}_f$ for some n . The smallest number n for which this happens is called the nilpotency class of \mathcal{C} .

Definition 5.19. [ENO2] A fusion category \mathcal{C} is called **weakly group-theoretical** if it is Morita equivalent to a nilpotent fusion category.

Theorem 5.20. *Every weakly group theoretical fusion category is a completely unitary fusion category.*

Proof. By Theorem 5.3, we only need to prove that every nilpotent fusion category is completely unitary.

Let \mathcal{C} be a nilpotent fusion category. We shall use induction on the nilpotency class of \mathcal{C} . If the nilpotency class is one, then \mathcal{C} is a pointed fusion category, so by Proposition 5.15, \mathcal{C} and $\mathcal{C} \boxtimes \mathcal{C}^{rev}$ are completely unitary fusion categories. Now, let \mathcal{C} be a nilpotent fusion category of nilpotency class n , so $\mathcal{C}_{ad} = \mathcal{C}^{(1)}$ has $n - 1$ nilpotency class and by hypothesis of induction \mathcal{C}_{ad} and $\mathcal{C}_{ad} \boxtimes (\mathcal{C}_{ad})^{rev}$ are completely unitary, thus by Theorem 5.17, $\mathcal{C} = \mathcal{C}^{(0)}$ is a completely unitary fusion category. \square

A (weak or quasi)-Hopf algebra is called weakly group-theoretical if $\text{Rep}(H)$ is a weakly group-theoretical fusion category.

Corollary 5.21. *Every weakly group-theoretical (quasi)-Hopf algebra is isomorphic to a (quasi)-Kac algebra.*

Proof. Let H be a weakly group-theoretical Hopf algebra. By Theorem 5.20, the fusion category $\text{Rep}(H)$ is equivalent to a unitary fusion category \mathcal{C} , and the forgetful functor defines a \mathcal{C} -module structure over Vec_f , so again by the complete unitarity of \mathcal{C} the fiber functor is equivalent to a unique exact $*$ -monoidal functor. By Tannaka-reconstruction theory for compact quantum groups (see [Wo]), the Hopf algebra H associated to a $*$ -monoidal fiber functor is isomorphic to a finite dimensional C^* -Hopf algebra, *i.e.*, a Kac algebra.

Now suppose that H is a weakly group-theoretical quasi-Hopf algebra. Then $\text{Rep}(H)$ is equivalent to a unique unitary fusion category \mathcal{C} . By [Mu1, Proposition 2.1] every unitary fusion category is a C^* -tensor category, so for every pair of objects X, Y , $\text{Hom}(X, Y)$ is a Hilbert space and $\langle fg, h \rangle = \langle g, f^*h \rangle$, $\langle fg, h \rangle = \langle f, hg^* \rangle$ for all morphisms in \mathcal{C} .

Let $R \in \mathcal{C}$ be the regular object, then $F(X) = \text{Hom}(R, X)$ defines a $*$ -functor $F : \mathcal{C} \rightarrow \text{Hilb}_f$. For every pair of simple objects $X_i, X_j \in \mathcal{C}$ there is a unitary isomorphism $J_{i,j} : F(X_i) \otimes_{\mathcal{C}} F(X_j) \rightarrow F(X_i \otimes X_j)$, that defines a quasi-fiber functor preserving the $*$ -structure with unitary constraint. Then by a standard reconstruction argument the algebra $\text{End}_{\mathcal{C}}(F)$ of functorial endomorphisms of F has a natural structure of quasi-Kac algebra, such that $\mathcal{U}\text{-Rep}(\text{End}_{\mathcal{C}}(F)) \cong \mathcal{C}$ as unitary fusion categories. \square

Remarks 5.22. (1) An analogous result to Corollary 5.21 is true for weakly group-theoretical weak Hopf algebras.

- (2) In the survey article [A], the following Question 7.8 is raised. Given a semisimple Hopf algebra H , does it admit a compact involution? Corollary 5.21 gives an affirmative answer for weakly group theoretical Hopf algebras.

It is not known (see [ENO2] Question 2) if there exist weakly integral fusion categories that are not weakly group-theoretical. Theorem 5.20 inspires the following **question**: *Is every weakly integral fusion category completely unitary or unitary?*

- Remarks 5.23.** (1) The answer is “no” without the weak integrality condition. In fact, every unitary fusion category is pseudo-unitary, (see [ENO1, Section 8.4]) but for example the Yang-Lee category is a non-integral and non pseudo-unitary fusion category. Indeed unitarity can fail in very dramatic ways, see [R1].
- (2) The question can be reduced to integral fusion categories. By [ENO1, Proposition 8.27] and [GeNik] for every weakly integral fusion category \mathcal{C} there is G -grading such that \mathcal{C}_e is an integral fusion category, so by Theorem 5.17, \mathcal{C} is completely unitary if and only if \mathcal{C}_e is completely unitary.

If \mathcal{C} is a unitary fusion category, the **unitary center** $\mathcal{Z}^*(\mathcal{C})$ is defined as the full fusion subcategory of the usual center $\mathcal{Z}(\mathcal{C})$, where $(X, c_{X,-}) \in \mathcal{Z}^*(\mathcal{C})$ if $c_{X,W} : X \otimes W \rightarrow W \otimes X$ are unitary natural transformations for all $W \in \mathcal{C}$. It is easy to see that $\mathcal{Z}^*(\mathcal{C})$ is a unitary fusion category. The following result appears in [Mu2, Theorem 6.4] we provide an alternate proof using our notation.

Proposition 5.24. *Let \mathcal{C} be a unitary fusion category. Then $\mathcal{Z}^*(\mathcal{C})$ is braided monoidally equivalent to $\mathcal{Z}(\mathcal{C})$.*

Proof. Let $\mathcal{C} \overline{\boxtimes} \mathcal{C}^{rev}$ be the external tensor product with the obvious structure of $*$ -fusion category, see Definition 5.4. The $*$ -category \mathcal{C} is a $\mathcal{C} \overline{\boxtimes} \mathcal{C}^{rev}$ -module $*$ -category. By [O2, Proposition 2.2] the center is equivalent to $\text{End}_{\mathcal{C} \overline{\boxtimes} \mathcal{C}^{rev}}(\mathcal{C})$ and it is easy to see that unitary center is $*$ -monoidally equivalent to $\text{End}_{\mathcal{C} \overline{\boxtimes} \mathcal{C}^{rev}}^*(\mathcal{C})$, so by Theorem 5.3 they are monoidally equivalent. \square

Corollary 5.25. *Let \mathcal{C} be a weakly group-theoretical braided fusion category. Then for every object $X \in \mathcal{C}$, the \mathcal{B}_n -representation on $\text{End}_{\mathcal{C}}(X^{\otimes n})$ is unitarizable. Moreover, if $\text{FPdim}(X) \in \mathbb{N}$ then the sequence $YB_{(X,\mathcal{C})}$ of algebras under $\mathbb{C}\mathcal{B}$ has a unitary quasi-localization of dimension $\text{FPdim}(X)$, and if \mathcal{C} has a fiber functor then $YB_{(X,\mathcal{C})}$ has a unitary localization of dimension $\text{FPdim}(X)$.*

Proof. We shall prove that every braided weakly group-theoretical fusion category is braided equivalent to a unitary braided fusion category.

The fusion category \mathcal{C} is weakly group-theoretical so it is a completely unitary fusion category. Since \mathcal{C} is braided, we have a canonical injective braided monoidal functor $F : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C})$, but by Proposition 5.24, $\mathcal{Z}(\mathcal{C})$ is braided equivalent to

the unitary center $\mathcal{Z}^*(\mathcal{C})$, so \mathcal{C} is braided equivalent to a unitary braided fusion category, *i.e.*, the braiding maps are unitary.

The others parts of the corollary follow from Proposition 4.17 and Corollary 5.21. \square

5.6. $\mathcal{C}(\mathfrak{sl}_3, 6)$: A case study. In this subsection we investigate a particular sequence of braid group representations that does not appear to have a localization, but does have both quasi- and (k, m) -localizations. This illustrates the criterion mentioned in Remark 4.18 and justifies the notion of (k, m) -localizations.

The integral unitary modular category $\mathcal{C}(\mathfrak{sl}_3, 6)$ has 10 simple objects and FP-dimension 36. It was shown in [NR] that $\mathcal{C}(\mathfrak{sl}_3, 6)$ is non-group-theoretical and in fact has minimal dimension among non-group-theoretical integral modular categories. The integrality of $\mathcal{C}(\mathfrak{sl}_3, 6)$ implies that there is a semisimple, finite dimensional quasi-triangular quasi-Hopf algebra A such that $\text{Rep}(A) \cong \mathcal{C}(\mathfrak{sl}_3, 6)$ as braided fusion categories. The simple object X analogous to the vector representation of \mathfrak{sl}_3 has $\text{FPdim}(X) = 2$ and tensor-generates $\mathcal{C}(\mathfrak{sl}_3, 6)$. By Jimbo's quantum Schur-Weyl duality ([Ji]), the unitary sequence of \mathcal{B}_n representations (ρ_X, W_n^X) is equivalent to the Jones-Wenzl representations factoring over the semisimple quotients $\mathcal{H}_n(3, 6)$ of the Hecke-algebras $\mathcal{H}_n(q)$ with $q = e^{2\pi i/3}$ (see [R3]). Explicitly, one has an isomorphism $\text{End}(X^{\otimes n}) \cong \mathcal{H}_n(3, 6)$ which intertwines the \mathcal{B}_n representations, and $\text{End}(X^{\otimes n})$ is generated by the image of the braid group. The eigenvalues of $\rho_X(\sigma_i)$ are -1 and $e^{2\pi i/6}$ in this case.

Moreover, (ρ_X, W_n^X) has a 2-dimensional unitary quasi-localization by Corollary 5.25. However, to explicitly determine the quasi-braided vector space (V, a, c) would require solving the pentagon and hexagon equations, a notoriously difficult task. The task would be significantly easier if $\mathcal{C}(\mathfrak{sl}_3, 6)$ were equivalent to $\text{Rep}(H)$ for some (strictly coassociative) Hopf algebra H since then one may assume a is trivial, *i.e.* (ρ_X, W_n^X) would have a 2-dimensional localization. This is not the case:

Lemma 5.26. *There is no unitary braided vector space of the form (V, c) with $\dim(V) = 2$ localizing (ρ_X, W_n^X) (in the sense of Definition 1.2).*

Proof. Dye [D, Theorem 4.1] has classified all 4×4 unitary solutions to the Yang-Baxter equation. Up to multiplying by a scalar and conjugation by matrices of the form $Q \otimes Q$ the solutions are of 4 forms. To see that none of these can localize (ρ_X, W_n^X) one need only check that the eigenvalues are not of the form $\{-\chi, \chi e^{2\pi i/6}\}$ with $\chi \in \mathbb{C}$. This is accomplished in [FRW, F]. \square

Applying our results we obtain the following:

Theorem 5.27. *Any semisimple quasi-Hopf algebra A with $\mathcal{C}(\mathfrak{sl}_3, 6) \cong \text{Rep}(A)$ is non-coassociative.*

Proof. Since $\text{FPdim}(\mathcal{C}(\mathfrak{sl}_3, 6)) = 36 = 2^2 3^2$, by [ENO2, Theorem 1.6] $\mathcal{C}(\mathfrak{sl}_3, 6)$ is solvable and hence weakly group-theoretical. Thus by Corollary 5.25 if $\mathcal{C}(\mathfrak{sl}_3, 6)$ admits a fiber functor (ρ_X, W_n^X) admits a unitary localization of dimension two and this contradicts Lemma 5.26. \square

Set $\zeta = e^{2\pi i/8}$ and consider the 8×8 unitary block-diagonal matrix:

$$(5.2) \quad R = \frac{-e^{-\pi i/3}}{\sqrt{2}} \begin{pmatrix} \zeta^{-1} & 0 & -\zeta^{-1} & 0 \\ 0 & \zeta & 0 & \zeta \\ \zeta & 0 & \zeta & 0 \\ 0 & -\zeta^{-1} & 0 & \zeta^{-1} \end{pmatrix} \oplus \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta & 0 & \zeta & 0 \\ 0 & \zeta^{-1} & 0 & -\zeta^{-1} \\ -\zeta^{-1} & 0 & \zeta^{-1} & 0 \\ 0 & \zeta & 0 & \zeta \end{pmatrix}$$

Then (\mathbb{C}^2, R) is a $(3, 1)$ -generalized braided vector space. In fact, we have the following:

Theorem 5.28. *(\mathbb{C}^2, R) with R as in (5.2) gives a $(3, 1)$ -localization of the sequence of braid group representations (ρ_X, W_n^X) .*

Proof. By the discussion above we may replace (ρ_X, W_n^X) by the Jones-Wenzl representation (π_n, V_n) associated with the semisimple quotient $\mathcal{H}_n(3, 6)$ of the specialized Hecke algebra $\mathcal{H}_n(q)$ with $q = e^{2\pi i/6}$ (see [W1]). Here the quotient is by the annihilator of the trace tr on $\mathcal{H}_n(q)$ uniquely determined by

- (1) $tr(1) = 1$
- (2) $tr(ab) = tr(ba)$
- (3) $tr(be_n) = tr(b)\eta$ where $b \in \mathcal{H}_{n-1}(q)$,

where e_n are the generators of $\mathcal{H}_n(q)$ and $\eta = \frac{1-q^{-2}}{1+q^3}$.

Thus it is enough to show that the \mathcal{B}_n -representation afforded by R in (5.2) factors over $\mathcal{H}_n(3, 6)$ and induces a faithful representation. Direct calculation shows that the $\rho_R(\sigma_i)$ indeed satisfy the defining relations of $\mathcal{H}_n(q)$ for $q = e^{2\pi i/6}$. Defining Tr on $\text{End}(\mathbb{C}^{2^{\otimes n+1}})$ (here $k + m(n-2) = n+1$) as $\frac{1}{2^{n+1}}$ times the usual trace, one concludes that $\rho_R^{-1}(Tr)(a) := Tr(\rho_R(a))$ coincides with tr by checking that it satisfies the relations above using standard techniques (see eg. [R3, proof of Lemma 3.1]). In particular we see that the kernel of ρ_R (restricted to $\mathcal{H}_n(q)$) lies in the annihilator of the trace tr so that ρ_R factors over the quotient $\mathcal{H}_n(3, 6)$. Faithfulness of the induced representation follows from a dimension count or by observing that Tr is faithful on $\text{End}((\mathbb{C}^2)^{\otimes n})$. \square

Remarks 5.29. (1) The matrix R above was derived from unpublished notes of Goldschmidt and Jones describing a sequence of quaternionic representations of \mathcal{B}_n . Indeed, let ι and j denote the usual generators of \mathbb{H} the quaternionic division algebra. Define

$$r = -e^{-\pi i/3}/2(1 + \iota \otimes j \otimes \iota + 1 \otimes (\iota j) \otimes 1 + j \otimes \iota \otimes \iota)$$

as an element of $\mathbb{H}^{\otimes 3}$. Applying the faithful 2-dimensional representation of \mathbb{H} to r yields the matrix R in (5.2).

- (2) The category $\mathcal{C}(\mathfrak{sl}_3, 6)$ is a subquotient of $\text{Rep}(U_q \mathfrak{sl}_3)$ with $q = e^{\pi i/6}$ with the 2-dimensional simple object X corresponding to the 3-dimensional vector representation V of $U_q \mathfrak{sl}_3$. As such, there is a 9×9 matrix solution \check{R} to the Yang-Baxter equation that associated to V (due to Jimbo [Ji]). However, \check{R} is not unitary and the braid group representations it affords has subrepresentations that are not subrepresentations of (ρ_X, W_n^X) .
- (3) The results of [FKW] imply that the representation spaces W_n^X can be uniformly embedded in a “local” Hilbert space but the braid group only acts on a small subspace. This issue was the main motivation for [RW].

6. CONJECTURES AND GENERAL RESULTS

We have the following version of [RW, Conjecture 4.1]:

Conjecture 6.1. *Let X be a simple object in a fusion category \mathcal{C} with (X, c) a Yang-Baxter operator. Then the following are equivalent:*

- (a) *The sequence of algebras $(YB_{(X,c)}, \rho^{(X,c)}, \iota)$ under \mathbb{CB} has a (unitary) generalized localization*
- (b) *The sequence of algebras $(YB_{(X,c)}, \rho^{(X,c)}, \iota)$ under \mathbb{CB} has a (unitary) quasi-localization*
- (c) $\text{FPdim}(X)^2 \in \mathbb{N}$
- (d) $\rho^{(X,c)}(\mathcal{B}_n)$ is a finite group for all n .

Remarks 6.2. (1) Corollary 5.25 shows that if \mathcal{C} is weakly group-theoretical and $\text{FPdim}(X) \in \mathbb{N}$ then (b) above holds. It is not known if there are any integral (or even weakly integral) fusion categories that are not weakly group-theoretical. If indeed \mathcal{C} integral implies \mathcal{C} weakly group-theoretical then $\text{FPdim}(X) \in \mathbb{N}$ implies (b), giving a slightly weaker version of (c) \Rightarrow (b). For this reason we parenthesize the word *unitary* as it is not unreasonable to expect that the conjecture is true with or without unitarity.

- (2) Conjecture 4.1 of [RW] is only concerned with simple objects in braided fusion categories for which the sequence of representations (ρ_n, W_n^X) is (unitarily) localizable. We are not aware of any counterexample to this more restrictive conjecture, but if the example discussed in Subsection 5.6 is such a counterexample then Conjecture 6.1 is an appropriate replacement.
- (3) If \mathcal{C} is group theoretical (and hence integral) it is known [ERW] that (d) holds for any object $X \in \mathcal{C}$ (for example if $\mathcal{C} \cong \text{Rep}(D^\omega G)$, where $D^\omega G$ is the twisted double of a finite group G). Integrality of \mathcal{C} and Prop. 4.17 imply that (b) and (c) also hold. The equivalence (c) \Leftrightarrow (d) has been considered elsewhere, see [J1, J4, GJ, FLW, LRW, R2, R3, NR].

The following version of [RW, Theorem 4.5] verifies part of Conjecture 6.1 under an additional hypothesis:

Theorem 6.3. *Let X be a simple object in a fusion category \mathcal{C} with c a Yang-Baxter operator on X such that*

- (a) $YB_{(X,c)} = YB_{(X,c)}$ (i.e. the image of \mathcal{B}_n generates $\text{End}(X^{\otimes n})$ as an algebra), and
- (b) the sequence of algebras $(YB_{(X,c)}, \rho^{(X,c)}, \iota)$ under \mathbb{CB} has either a generalized or quasi-localization

then $\text{FPdim}(X)^2 \in \mathbb{N}$.

Proof. By passing to fusion subcategories, we may assume X is a tensor generator for \mathcal{C} .

For quasi-localization Corollary 4.14 implies that the same proof as in [RW, Theorem 4.5] goes through without change as quasi-localizability implies the same combinatorial consequences as localizability.

For the case of generalized localization we adapt the proof in [RW, Theorem 4.5]. Suppose (W, γ) is a (k, m) -localization of $(YB_{(X,c)}, \rho^{(X,c)}, \iota)$ with injective morphism $\phi : YB_{(X,c)} \rightarrow gYB_{(W,\gamma)}$. Observe that $\text{End}_{gYB_{(W,\gamma)}} = \text{End}_{\mathbb{C}}(W^{\otimes k+m(n-2)})$ and set $w = \dim(W)$. Fix an ordering $X_0 = \mathbf{1}, X_1, \dots, X_{N-1}$ of the simple objects and define $H_i^n = \text{Hom}(X_i, X^{\otimes n})$ for each i with $\text{Hom}(X_i, X^{\otimes n}) \neq 0$. By (a), the H_i^n form a complete set of simple $\rho^{(X,c)}(\mathbb{CB}_n)$ -modules. Let N_X denote the fusion matrix of X , i.e. such that $(N_X)_{i,j} = \dim \text{Hom}(X_j, X \otimes X_i)$. Let $l \in \mathbb{N}$ be minimal with the property that for all $i = 0, \dots, N-1$ there exists an $s \leq l$ such that $H_i^s \neq 0$ (l exists by [DGNO, Lemma F.2]). Now let $p \geq 1$ be minimal such that $H_0^p \neq 0$ so that N_X is an irreducible matrix of period p . Denote by G_n the inclusion matrix for the algebras $\text{End}(X^{\otimes n}) \subset \text{End}(X^{\otimes n+1})$, i.e. the matrix of multiplicities obtained by restricting the H_j^{n+1} to $\text{End}(X^{\otimes n})$ and decomposing into a direct sum of H_i^n . Semisimplicity of $\text{End}(X^{\otimes n})$ and the injectivity of ϕ imply that $W^{\otimes k+m(n-2)} \cong \bigoplus_i \mu_i^n H_i^n$ as $\text{End}(X^{\otimes n})$ -modules for some $\mu_i^n > 0$. Define a vector of multiplicities $(\mathbf{a}_n)_i := \mu_i^n$. The commutativity of the diagram in Definition 4.5 and the injectivity of ϕ imply:

$$(6.1) \quad w^m \mathbf{a}_n = G_n \mathbf{a}_{n+1}$$

for all $n \geq l$.

The (square!) inclusion matrices $G^{(i)} = \prod_{j=0}^{p-1} G_{l+i+j}$ of $\text{End}(X^{\otimes l+i}) \subset \text{End}(X^{\otimes l+i+p})$ are primitive (as N_X is irreducible of period p) and hence the Perron-Frobenius eigenvalue of each $G^{(i)}$ is simple. We will first show that \mathbf{a}_l is a Perron-Frobenius eigenvector for $G := G^{(0)}$. Eqn. (6.1) implies that

$$(w^m)^p \mathbf{a}_l = G \mathbf{a}_{l+p}.$$

For simplicity, let us define $\alpha_0 := \mathbf{a}_l$, $\alpha_n := \mathbf{a}_{l+pn}$ and $M = (w^m)^p$. In this notation, the above equation implies

$$M^n \alpha_0 = G^n \alpha_n$$

for all $n \geq 0$. Let Λ denote the Perron-Frobenius eigenvalue of G and let \mathbf{v} and \mathbf{w} be positive right and left eigenvectors such that $\mathbf{w}\mathbf{v} = 1$ and $\lim_{s \rightarrow \infty} (\frac{1}{\Lambda} G)^s = \mathbf{v}\mathbf{w}$. We can rewrite the above equation as follows:

$$\alpha_0 = \frac{\Lambda^n}{M^n} \left(\frac{1}{\Lambda} G \right)^n \alpha_n.$$

By taking a limit on the right-hand-side and then applying the property of the eigenvectors \mathbf{v} and \mathbf{w} we have:

$$\alpha_0 = \lim_{n \rightarrow \infty} \frac{\Lambda^n}{M^n} \mathbf{v}\mathbf{w} \alpha_n = \lim_{n \rightarrow \infty} \left(\frac{\Lambda^n \mathbf{w} \alpha_n}{M^n} \right) \mathbf{v}.$$

The limit exists and thus $\alpha_0 = \mathbf{a}_l$ is an eigenvector for G as a non-zero multiple of \mathbf{v} . Similarly, each α_j is also a Perron-Frobenius eigenvector for G . The integrality of G and α_0 implies that Λ is rational and, moreover, the eigenvalues of G are algebraic integers, thus Λ is a (rational) integer. The same argument shows that each \mathbf{a}_{l+i} is a Perron-Frobenius eigenvector for $G^{(i)}$. Now we will show that $\text{FPdim}(X)^2 \in \mathbb{N}$. As N_X is irreducible with period p we may reorder the simple objects X_0, \dots, X_{N-1} so that $(N_X)^p$ is block diagonal with primitive blocks $G^{(i)}$. Let us denote the Perron-Frobenius eigenvalue of N_X by λ . Then by the Frobenius-Perron theorem each $G^{(i)}$ has λ^p as its Perron-Frobenius eigenvalue which is a (rational) integer by the above arguments. But λ must reside in an abelian (cyclotomic) extension of \mathbb{Q} and this implies that $\lambda^s \in \mathbb{Z}$ for some $s \leq 2$. \square

REFERENCES

- [A] N. Andruskiewitsch, *About finite dimensional Hopf algebras*. Quantum symmetries in theoretical physics and mathematics (Bariloche, 2000), Contemp. Math., 294, 157, Amer. Math. Soc., Providence, RI, 2002.
- [AS] N. Andruskiewitsch and H.-J. Schneider, *Pointed Hopf algebras*. New directions in Hopf algebras, 1–68, Math. Sci. Res. Inst. Publ., 43, Cambridge Univ. Press, Cambridge, 2002.
- [B] J. Baez, *Higher-Dimensional Algebra II. 2-Hilbert Spaces*, Adv. Math. **127** (1997), 125–189.
- [BK] B. Bakalov; A. Kirillov, Jr., *Lectures on Tensor Categories and Modular Functors*, University Lecture Series, vol. **21**, Amer. Math. Soc., 2001.
- [BNSz] G. Böhm, F. Nill and K. Szlachányi, *Weak Hopf algebras I. Integral theory and C^* -structure*, J. Algebra, 221 (1999), 385–438.
- [BSz] G. Böhm and K. Szlachányi, *Weak Hopf Algebras: II. Representation Theory, Dimensions, and the Markov Trace*, J. Algebra, 233 (2000), 156–212.
- [DGNO] V. Drinfeld, S. Gelaki, D. Nikshych and V. Ostrik, *On Braided Fusion Categories I*, Selecta Math. N.S. **16**, 1 (2010) 1–119.
- [Dav] A. Davydov, *Nuclei of categories with tensor products*. Theory and Applications of Categories, Vol. 18, No. 16, (2007), 440–472.

- [D] H. Dye, *Unitary solutions to the Yang-Baxter equation in dimension four*. Quantum information processing **2** (2002) nos. 1-2, 117–150 (2003).
- [ENO1] P. Etingof, D. Nikshych and V. Ostrik, *On fusion categories*. Ann. of Math. (2) 162 (2005), no. 2, 581–642.
- [ENO2] P. Etingof, D. Nikshych and V. Ostrik, *Weakly group-theoretical and solvable fusion categories*, Adv. Math. 226 (2011) 176–205.
- [ENO3] P. Etingof, D. Nikshych and V. Ostrik, *Fusion categories and homotopy theory*, Quantum Topol. 1 (3) (2010) 209–273. Preprint arXiv:0909.3140.
- [ERW] P. Etingof, E. C. Rowell and S. J. Witherspoon, *Braid group representations from quantum doubles of finite groups*. Pacific J. Math. **234** (2008) no. 1, 33–41.
- [FRW] J. Franko, E. C. Rowell and Z. Wang, *Extraspecial 2-groups and images of braid group representations*, J. Knot Theory Ramifications **15** (2006) no. 4, 1–15.
- [F] J. Franko, *Braid group representations arising from the Yang-Baxter equation*. J. Knot Theory Ramifications, **19** (2010), no. 4, 525–538.
- [FKW] M. H. Freedman; A. Kitaev; Z. Wang, *Simulation of topological field theories by quantum computers*. Comm. Math. Phys. **227** (2002) no. 3, 587–603.
- [FLW] M. H. Freedman, M. J. Larsen and Z. Wang, *The two-eigenvalue problem and density of Jones representation of braid groups*. Comm. Math. Phys. 228 (2002), 177–199.
- [Ga1] C. Galindo, *Clifford theory for tensor categories*, J. Lond. Math. Soc., (2) 83 (2011) 57–78.
- [Ga2] C. Galindo, *Clifford theory for graded fusion categories*, preprint arXiv:1010.5283 .
- [Ga3] C. Galindo, *Crossed product tensor categories*, J. Algebra, 337 (2011) 233–252.
- [GeNik] S. Gelaki and D. Nikshych, *Nilpotent fusion categories*, Adv. Math., **217** (2008), 1053–1071.
- [GJ] D. M. Goldschmidt and V. F. R. Jones, *Metaplectic link invariants*. Geom. Dedicata **31** (1989) no. 2, 165–191.
- [GHJ] F. M. Goodman, P. de la Harpe, V. F. R. Jones, *Coxeter graphs and towers of algebras*. Mathematical Sciences Research Institute Publications, 14. Springer-Verlag, New York, 1989. x+288 pp.
- [Gr] J. Greenough, *Monoidal 2-structure of Bimodule Categories*, J. Algebra, 324 (2010), 1818–1859. Preprint arXiv:0911.4979.
- [Ji] M. Jimbo, *A q -analogue of $U(\mathfrak{gl}(N+1))$, Hecke algebra, and the Yang-Baxter equation*. Lett. Math. Phys. **11** (1986), no. 3, 247–252.
- [J1] V. F. R. Jones, *Braid groups, Hecke algebras and type II_1 factors*, Geometric methods in operator algebras (Kyoto, 1983), 242–273, Pitman Res. Notes Math. Ser. **123**, Longman Sci. Tech., Harlow, 1986.
- [J4] V. F. R. Jones, *On a certain value of the Kauffman polynomial*, Comm. Math. Phys. **125** (1989), 459–467.
- [JS] A. Joyal and R. Street, *Braided Tensor Categories*, Adv. Math. 102, (1993), 20–78.
- [Ks] C. Kassel, *Quantum Groups*. Graduate Texts in Mathematics **155**, Springer-Verlag, New York, 1995.
- [LRW] M. J. Larsen, E. C. Rowell and Z. Wang: *The N -eigenvalue problem and two applications*. Int. Math. Res. Not. **2005** (2005) no. 64, 3987–4018.
- [Mu1] M. Mueger, *Galois theory for braided tensor categories and the modular closure*, Adv. Math. **150** (2000), 151–201.
- [Mu2] M. Mueger, *From subfactors to categories and topology II. The quantum double of tensor categories and subfactors*, J. Pure Appl. Alg. **180** (2003), 159–219.
- [NR] D. Naidu and E. C. Rowell, *A finiteness property for braided fusion categories*, to appear in Algebr. Represent. Theory.

- [O1] V. Ostrik, *Module categories, Weak Hopf Algebras and Modular invariants.*, Transform. Groups. **8** (2003), 177–206.
- [O2] V. Ostrik, *Module categories over the Drinfeld double of a finite group*, Int. Math. Res. Not. (2003) no. 27, 1507–1520.
- [Rot] J. Rotman, *An Introduction to Homological Algebra*, Academic Press, New York-San Francisco-London, (1979).
- [R1] E. C. Rowell *On a family of non-unitarizable ribbon categories*, Math. Z. 250 (2005), no. 4, 745–774.
- [R2] E. C. Rowell, *Braid representations from quantum groups of exceptional Lie type*, Rev. Un. Mat. Argentina **51** (2010) no. 1, 165–175.
- [R3] E. C. Rowell, *A quaternionic braid representation (after Goldschmidt and Jones)*, to appear in Quant. Topol. arXiv:1006.4808.
- [RSW] E. Rowell, R. Stong and Z. Wang, *On classification of modular tensor categories*, Comm. Math. Phys. **292** (2009), no. 2, 343–389.
- [RW] E. C. Rowell and Z. Wang, *Localization of unitary braid group representations*, preprint arXiv:1009.0241
- [RZWG] E. C. Rowell, Y. Zhang, Y.-S. Wu and M.-L. Ge, *Extraspecial two-groups, generalized Yang-Baxter equations and braiding quantum gates*, Quantum Inf. Comput. **10** (2010) no. 7-8, 0685–0702.
- [Tam] D. Tambara, *Invariants and semi-direct products for finite group actions on tensor categories*, J. Math. Soc. Japan **53** (2001), 429–456.
- [TW] V. G. Turaev; H. Wenzl, *Semisimple and modular tensor categories from link invariants*, Math. Ann. **309** (1997), 411–461.
- [W1] H. Wenzl, *Hecke algebras of type A_n and subfactors*. Invent. Math. **92** (1988) no. 2, 349–383.
- [Wo] S.L. Woronowicz, *Tannaka-Krein duality for compact matrix pseudogroups. Twisted $SU(N)$ groups*. Invent. Math. **93**, (1988) 35–76.

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE LOS ANDES, CARRERA 1 N. 18A - 10, BOGOTÁ, COLOMBIA
E-mail address: `cn.galindo1116@uniandes.edu.co`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TOLEDO, TOLEDO, OH 43606-3390
E-mail address: `seungmoon.hong@utoledo.edu`

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843-3368
E-mail address: `rowell@math.tamu.edu`